

DECOMPOSITION MATRICES OF BIRMAN-MURAKAMI-WENZL ALGEBRAS

HEBING RUI AND LINLIANG SONG

ABSTRACT. In this paper, we calculate decomposition matrices of the Birman-Murakami-Wenzl algebras over \mathbb{C} .

1. INTRODUCTION

One of key problems in studying structure of a finite dimensional algebra is to determine its semisimple quotient. This leads to calculate dimensions of its simple modules. In this paper, we address this problem on a Birman-Murakami-Wenzl algebra over \mathbb{C} by determining its decomposition numbers.

Recall that Birman-Murakami-Wenzl algebras are unital associative R -algebras introduced in [6, 20], where R is a commutative ring containing 1 and invertible elements ϱ, q and $q - q^{-1}$. Suppose R is a field κ . If $\varrho \notin \{q^a, -q^a \mid a \in \mathbb{Z}\}$, Rui and Si [24] proved that $\mathcal{B}_r(\varrho, q)$ is Morita equivalent to $\bigoplus_{i=0}^{\lfloor r/2 \rfloor} \mathcal{H}_{r-2i}$ where \mathcal{H}_{r-2i} are Hecke algebras associated to symmetric groups \mathfrak{S}_{r-2i} . In non-semisimple cases and $\kappa = \mathbb{C}$, by Ariki's result on decomposition numbers of Hecke algebras in [4], decomposition numbers of $\mathcal{B}_r(\varrho, q)$ are determined by the values of certain inverse Kazhdan-Lusztig polynomials at $q = 1$ associated to some extended affine Weyl groups of type A . If $\varrho \in \{q^a, -q^a\}$ for some $a \in \mathbb{Z}$ and if q^2 is not a root of unity, Rui and Si classified blocks of $\mathcal{B}_r(\varrho, q)$ over κ [22]. Via such results together with Martin's arguments on the decomposition matrices of Brauer algebras over \mathbb{C} in [19], Xu showed that $\mathcal{B}_r(\varrho, q)$ is multiplicity-free over \mathbb{C} [31]. In other words, the multiplicity of a simple module in a cell (or standard) $\mathcal{B}_r(\varrho, q)$ -module is either 1 or 0 if κ is \mathbb{C} .

The aim of this paper is to calculate decomposition matrices of $\mathcal{B}_r(\varrho, q)$ over \mathbb{C} when $\varrho \in \{-q^a, q^a\}$ for some $a \in \mathbb{Z}$ and q^2 is a root of unity. In this case, it is enough to assume either $\varrho = -q^{2n+1}$ or $\varrho = q^n$ for some $n \in \mathbb{Z}$ such that $n \gg 0$. In the first case, Hu [15] proved that there is an integral Schur-Weyl duality between $\mathcal{B}_r(-q^{2n+1}, q)$ and the quantum group $\mathbf{U}(\mathfrak{sp}_{2n})$ associated to \mathfrak{sp}_{2n} . In particular, Hu proved that $\mathcal{B}_r(-q^{2n+1}, q)$ is isomorphic to $\text{End}_{\mathbf{U}(\mathfrak{sp}_{2n})}(V^{\otimes r})$ if $n \geq r$, where V is the natural representation of $\mathbf{U}(\mathfrak{sp}_{2n})$. Moreover, Hu's arguments in [15] can be used smoothly to prove that $\mathcal{B}_r(q^n, q)$ is isomorphic to $\text{End}_{\mathbf{U}(\mathfrak{g})}(V^{\otimes r})$ if $\lfloor \frac{n+1}{2} \rfloor > r$, where V is the natural representation of $\mathbf{U}(\mathfrak{so}_{n+1})$. Motivated by our work on quantized walled Brauer algebras in [25], we establish an explicit relationship between decomposition numbers of $\mathcal{B}_r(\varrho, q)$ with $\varrho \in \{-q^{2n+1}, q^n\}$ and the multiplicities of Weyl modules in indecomposable direct summands of $V^{\otimes r}$ (called partial tilting modules). When the ground field is \mathbb{C} and e , the order of q^2 is bigger than 29, such multiplicities have been

Rui is supported by NSFC (grant no. 11025104).

given in [26]¹. Suppose $e = \infty$. By arguments similar to those in [11], the decomposition matrices of $\mathcal{B}_r(\varrho, q)$ are the same as those for $\mathcal{B}_r(\varrho, q)$ with $e \gg 0$. In particular, we recover [29, Theorem 5.6] by assuming that $\varrho = -q^{2n+1}$.

We organize this paper as follows. In section 2, after recalling some well known results on quantum groups, we use Hu's arguments in [15] to show that $\mathcal{B}_r(q^n, q)$ is isomorphic to $\text{End}_{\mathbf{U}(\mathfrak{so}_{n+1})}(V^{\otimes r})$ if $\lfloor \frac{n+1}{2} \rfloor > r$, where V is the natural representation of $\mathbf{U}(\mathfrak{so}_{n+1})$. In section 3, we prove that $V^{\otimes r}$ is self-dual as $(\mathbf{U}(\mathfrak{g}), \mathcal{B}_r(\varrho, q))$ -bimodule where $\mathfrak{g} \in \{\mathfrak{sp}_{2n}, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$ and ϱ is given in (2.20). In section 4, we classify highest weight vectors of $V^{\otimes r}$. This leads us to establish an explicit relationship between decomposition numbers of $\mathcal{B}_r(\varrho, q)$ with some special parameters ϱ in (2.20) and the multiplicities of Weyl modules in indecomposable tilting modules for $\mathbf{U}(\mathfrak{g})$. So, we can use Soergel's results in [26, 27] to calculate decomposition numbers of Birman-Murakami-Wenzl algebras over \mathbb{C} . Together with some previous results, we settle the problem on decomposition matrices of $\mathcal{B}_r(\varrho, q)$ over \mathbb{C} under the assumption $e \geq 29$.

2. SCHUR-WEYL DUALITY IN CLASSICAL TYPES

Throughout, let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with the quotient field $\mathbb{Q}(v)$ where v is an indeterminate. For any $n \in \mathbb{N}$, let

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}. \quad (2.1)$$

For $m, n, d \in \mathbb{N}$, following [21], define

$$[n]_d! := \prod_{i=1}^n \frac{v^{di} - v^{-di}}{v^d - v^{-d}}, \quad \left[\begin{matrix} m+n \\ n \end{matrix} \right]_d = \frac{[m+n]_d!}{[m]_d! [n]_d!} \in \mathcal{A}. \quad (2.2)$$

The Cartan matrix is an $n \times n$ matrix $A = (a_{ij})$ with entries $a_{ij} \in \mathbb{Z}$, $1 \leq i, j \leq n$ such that $(d_i a_{ij})$ is symmetric and positive definite, where $d_i \in \{1, 2, 3\}$ and $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$. The quantum group U_v associated with A is the associative $\mathbb{Q}(v)$ -algebra generated by $\{e_i, f_i, k_i^{\pm 1} \mid 1 \leq i \leq n\}$ subject to the relations:

$$\left\{ \begin{array}{l} k_i k_i^{-1} = 1 = k_i^{-1} k_i, \quad k_i k_j = k_j k_i, \\ k_i e_j k_i^{-1} = v^{d_i a_{ij}} e_j, \\ k_i f_j k_i^{-1} = v^{-d_i a_{ij}} f_j, \\ e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{v^{d_i} - v^{-d_i}}, \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} e_i^{1-a_{ij}-s} e_j e_i^s = 0, \text{ if } i \neq j, \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} f_i^{1-a_{ij}-s} f_j f_i^s = 0, \text{ if } i \neq j, \end{array} \right. \quad (2.3)$$

where δ_{ij} is the Kronecker delta. It is known that U_v is a Hopf algebra with the comultiplication Δ , counit ϵ and antipode S defined by

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i, & \epsilon(e_i) &= 0, & S(e_i) &= -k_i^{-1} e_i, \\ \Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i, & \epsilon(f_i) &= 0, & S(f_i) &= -f_i k_i, \\ \Delta(k_i) &= k_i \otimes k_i, & \epsilon(k_i) &= 1, & S(k_i) &= k_i^{-1}. \end{aligned} \quad (2.4)$$

¹Soergel needs the equivalence of categories between modules for quantum groups at roots of unity and corresponding module categories for Kac-Moody algebras in [27]. Due to [17], this equivalence is only proved when $e \geq 29$. Thanks Professor H.H. Andersen for his explanation.

For all positive integers k , following [21], let

$$e_i^{(k)} = e_i^k / [k]_{d_i}!, \text{ and } f_i^{(k)} = f_i^k / [k]_{d_i}!. \quad (2.5)$$

Then U_v contains the \mathcal{A} -subalgebra \mathbf{U} generated by $\{e_i^{(k)}, f_i^{(k)}, k_i^{\pm 1} \mid 1 \leq i \leq n, k \in \mathbb{Z}^{>0}\}$. Further, \mathbf{U} is a Hopf algebra such that comultiplication, counit and antipode are obtained from those for U_v by restrictions.

In this paper, we consider quantum groups associated with complex semisimple Lie algebras $\mathfrak{g} \in \{\mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}\}$. According to [7], we have the root systems for \mathfrak{g} so that $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are orthonormal and if $\mathfrak{g} = \mathfrak{sl}_{n+1}$, also include ϵ_{n+1} . Let $\Pi = \{\alpha_i \mid 1 \leq i \leq n\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$ and

$$\alpha_n = \begin{cases} \epsilon_n - \epsilon_{n+1}, & \text{if } \mathfrak{g} = \mathfrak{sl}_{n+1}, \\ \epsilon_n, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1}, \\ 2\epsilon_n, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ \epsilon_{n-1} + \epsilon_n, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}. \end{cases} \quad (2.6)$$

Then Π is a set of simple roots associated with \mathfrak{g} . The weight lattice P is $\oplus_{i=1}^n \mathbb{Z}\omega_i$, where ω_i 's are fundamental weights given by

- (1) $\omega_i = \epsilon_1 + \dots + \epsilon_i - \frac{i}{n+1}(\epsilon_1 + \dots + \epsilon_{n+1})$, $1 \leq i \leq n$ if $\mathfrak{g} = \mathfrak{sl}_{n+1}$,
- (2) $\omega_i = \epsilon_1 + \dots + \epsilon_i$, $1 \leq i \leq n-1$, and $\omega_n = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_n)$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$,
- (3) $\omega_i = \epsilon_1 + \dots + \epsilon_i$, $1 \leq i \leq n$, if $\mathfrak{g} = \mathfrak{sp}_{2n}$,
- (4) $\omega_i = \epsilon_1 + \dots + \epsilon_i$, $1 \leq i \leq n-2$, $\omega_{n-1} = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n)$ and $\omega_n = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_n)$ if $\mathfrak{g} = \mathfrak{so}_{2n}$.

Let $P^+ = \oplus_{i=1}^n \mathbb{N}\omega_i$. Then P^+ is the set of all dominant integral weights. For $\alpha_i, \alpha_j \in \Pi$, let

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j) / (\alpha_i, \alpha_i),$$

where $(\ , \)$ is the symmetric bilinear form such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$. The Cartan matrix A associated with \mathfrak{g} is the $n \times n$ matrix (a_{ij}) , which is the transpose of that in [7]. So, the quantum groups $U_v(\mathfrak{g})$ associated with \mathfrak{g} defined in (2.3) are the same as those in [14]. They are associative algebras over $\mathbb{Q}(v)$ such that $v = q^{1/2}$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $v = q$, otherwise. Further,

- (1) $d_i = 1$, $1 \leq i \leq n$ if $\mathfrak{g} \in \{\mathfrak{sl}_{n+1}, \mathfrak{so}_{2n}\}$,
- (2) $d_i = 2$, $1 \leq i \leq n-1$ and $d_n = 1$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$,
- (3) $d_i = 1$, $1 \leq i \leq n-1$ and $d_n = 2$ if $\mathfrak{g} = \mathfrak{sp}_{2n}$.

If M is a $U_v(\mathfrak{g})$ -module, let

$$M_\lambda = \{m \in M \mid k_i m = v_i^{\langle \lambda, \alpha_i \rangle} m, 1 \leq i \leq n\}, \text{ for any } \lambda \in P, \quad (2.7)$$

where $v_i = v^{d_i}$ and $\langle \lambda, \alpha_i \rangle = 2(\lambda, \alpha_i) / (\alpha_i, \alpha_i)$. Then M_λ is called the weight space of M with respect to the weight λ if $M_\lambda \neq 0$. For any field κ which is an \mathcal{A} -algebra, let $\mathbf{U}_\kappa(\mathfrak{g}) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathcal{A}} \kappa$, where $\mathbf{U}(\mathfrak{g})$ is the \mathcal{A} -form of $U_v(\mathfrak{g})$. If M is a $\mathbf{U}_\kappa(\mathfrak{g})$ -module, the weight space of M can be defined by base change. Later on, we write

$$\text{wt}(m) = \lambda \text{ if } m \in M_\lambda.$$

In the remaining part of this paper, we always assume that

$$N = \begin{cases} n, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ 2n+1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1}, \\ 2n, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}. \end{cases} \quad (2.8)$$

If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we write $i' = 2n+2-i$, $1 \leq i \leq n+1$, and hence

$$1 < 2 < \cdots < n < n+1 < n' < \cdots < 1'. \quad (2.9)$$

If $\mathfrak{g} \in \{\mathfrak{sp}_{2n}, \mathfrak{so}_{2n}\}$, we write $i' = 2n-i+1$, $1 \leq i \leq n$, and hence

$$1 < 2 < \cdots < n < n' < \cdots < 1'. \quad (2.10)$$

In any case, we set $k'' = k$ for any $1 \leq k \leq n$. If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $(n+1)' = n+1$. Unless otherwise state, we always assume that κ is a field which is an \mathcal{A} -algebra such that v acts on κ via $q \in \kappa^*$ (resp., $q^{1/2} \in \kappa^*$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$).

Lemma 2.1. *Let $V = \bigoplus_{i=1}^N \kappa v_i$, where N is given in (2.8). Then V is a left $\mathbf{U}_\kappa(\mathfrak{g})$ -module such that the following conditions hold.*

- (1) *If $\mathfrak{g} = \mathfrak{sl}_n$, then*
 - (a) $e_i v_k = \delta_{k,i+1} v_i$,
 - (b) $f_i v_k = \delta_{i,k} v_{i+1}$,
 - (c) $k_i v_k = q^\epsilon v_k$, where $\epsilon = 1$ (resp., -1) if $k = i$ (resp., $i+1$) and $\epsilon = 0$ in the remaining cases.
- (2) *If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then for $i \neq n$,*
 - (a) $e_i v_{i+1} = v_i$, $e_i v_{i'} = -v_{(i+1)'}$ and $e_i v_k = 0$, otherwise,
 - (b) $f_i v_i = v_{i+1}$, $f_i v_{(i+1)'} = -v_{i'}$, and $f_i v_k = 0$ otherwise,
 - (c) $k_i v_k = q v_k$ (resp., $q^{-1} v_k$) if $k \in \{i, (i+1)'\}$ (resp., $k \in \{i+1, i'\}$), and $k_i v_k = v_k$, otherwise,
 - (d) $e_n v_{n+1} = v_n$, $e_n v_{n'} = -q^{-1/2} v_{n+1}$, and $e_n v_k = 0$, otherwise,
 - (e) $f_n v_n = [2]_{q^{1/2}} v_{n+1}$, $f_n v_{n+1} = -q^{1/2} [2]_{q^{1/2}} v_{n'}$ and $f_n v_k = 0$, otherwise,
 - (f) $k_n v_n = q v_n$, $k_n v_{n'} = q^{-1} v_{n'}$ and $k_n v_k = v_k$, otherwise.
- (3) *If $\mathfrak{g} = \mathfrak{sp}_{2n}$, then for $i \neq n$,*
 - (a) $e_i v_k$, $f_i v_k$ and $k_i v_k$ satisfy the formulae in (2a)–(2c), respectively,
 - (b) $e_n v_{n'} = v_n$ and $e_n v_k = 0$, otherwise,
 - (c) $f_n v_n = v_{n'}$ and $f_n v_k = 0$, otherwise,
 - (d) $k_n v_n = q^2 v_n$, $k_n v_{n'} = q^{-2} v_{n'}$ and $k_n v_k = v_k$, otherwise.
- (4) *If $\mathfrak{g} = \mathfrak{so}_{2n}$, then for $i \neq n$,*
 - (a) $e_i v_k$, $f_i v_k$ and $k_i v_k$ satisfy the formulae in (2a)–(2c), respectively,
 - (b) $e_n v_{n'} = v_{n-1}$, $e_n v_{(n-1)'} = -v_n$ and $e_n v_k = 0$, otherwise,
 - (c) $f_n v_{n-1} = v_{n'}$, $f_n v_n = -v_{(n-1)'}$ and $f_n v_k = 0$, otherwise,
 - (d) $k_n v_k = q v_k$ (resp., $q^{-1} v_k$) if $k \in \{n-1, n\}$ (resp., $\{(n-1)', n'\}$), and $k_n v_k = v_k$, otherwise.

Proof. When $\kappa = \mathbb{Q}(v)$, Lemma 2.1(1)–(4) have been given in [14, (4.16)]². In general, since V has a \mathcal{A} -lattice spanned by $\{v_i \mid 1 \leq i \leq N\}$, which is a left $\mathbf{U}(\mathfrak{g})$ -module, the result follows from arguments on base change. \square

The κ -space V in Lemma 2.1 is known as the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$.

Corollary 2.2. *Let $V = \bigoplus_{i=1}^N \kappa v_i$ be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$.*

- (1) *If $\mathfrak{g} = \mathfrak{sl}_n$, then $\text{wt}(v_i) = \epsilon_i - \frac{1}{n} \sum_{i=1}^n \epsilon_i$.*
- (2) *If $\mathfrak{g} = \{\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}\}$ then $\text{wt}(v_i) = -\text{wt}(v_{i'}) = \epsilon_i$, $1 \leq i \leq n$. Further, if $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then $\text{wt}(v_{n+1}) = 0$.*

Proof. The result follows from Lemma 2.1, immediately. \square

In the remaining part of this paper, we always assume that

$$\rho = \begin{cases} (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, \frac{3}{2} - n, \frac{1}{2} - n), & \mathfrak{g} = \mathfrak{so}_{2n+1}, \\ (n - 1, n - 2, \dots, 1, 0, 0, -1, \dots, 2 - n, 1 - n), & \mathfrak{g} = \mathfrak{so}_{2n}, \\ (n, n - 1, \dots, 1, -1, \dots, 1 - n, -n), & \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases} \quad (2.11)$$

Corollary 2.3. *Let $V = \bigoplus_{i=1}^N \kappa v_i$ be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$. The 1-dimensional κ -subspace of $V^{\otimes 2}$ generated by $\alpha = \sum_{k=1}^N q^{\rho_{k'}} \varepsilon_{k'} v_k \otimes v_{k'}$ is a left $\mathbf{U}_\kappa(\mathfrak{g})$ -module where ρ is defined in (2.11) and $\varepsilon_i = 1$ unless $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $n + 1 \leq i \leq 2n$. In the later case, $\varepsilon_i = -1$.*

Proof. Obviously, $k_i \alpha = \alpha$, $1 \leq i \leq n$. Suppose $1 \leq i \leq n - 1$. By Lemma 2.1,

$$\begin{aligned} e_i \alpha &= q^{\rho_{(i+1)'}} \varepsilon_{(i+1)'} e_i v_{i+1} \otimes v_{(i+1)'} + q^{\rho_i} \varepsilon_i e_i v_{i'} \otimes v_i \\ &\quad + q^{\rho_{i'}} \varepsilon_{i'} k_i v_i \otimes e_i v_{i'} + q^{\rho_{i+1}} \varepsilon_{i+1} k_i v_{(i+1)'} \otimes e_i v_{i+1} \\ &= (q^{\rho_{(i+1)'}} \varepsilon_{(i+1)'} - q^{\rho_{i'}+1} \varepsilon_{i'}) v_i \otimes v_{(i+1)'} + (q^{\rho_{i+1}+1} \varepsilon_{i+1} - q^{\rho_i} \varepsilon_i) v_{(i+1)'} \otimes v_i, \end{aligned}$$

Since $q^{\rho_{(i+1)'}} \varepsilon_{(i+1)'} - q^{\rho_{i'}+1} \varepsilon_{i'} = q^{\rho_{i+1}+1} \varepsilon_{i+1} - q^{\rho_i} \varepsilon_i = 0$, $1 \leq i \leq n - 1$, $e_i \alpha = 0$.

If $\mathfrak{g} = \mathfrak{sp}_{2n}$, by Lemma 2.1,

$$e_n \alpha = q^{\rho_n} \varepsilon_n v_{n'} \otimes v_n + q^{\rho_{n'}} \varepsilon_{n'} k_n v_n \otimes e_n v_{n'} = q v_n \otimes v_n - q v_n \otimes v_n = 0.$$

If $\mathfrak{g} = \mathfrak{so}_{2n}$, by Lemma 2.1,

$$\begin{aligned} e_n \alpha &= q^{\rho_n} e_n v_{n'} \otimes v_n + q^{\rho_{n-1}} e_n v_{(n-1)'} \otimes v_{n-1} + q^{\rho_{n'}} k_n v_n \otimes e_n v_{n'} \\ &\quad + q^{\rho_{(n-1)'}} k_n v_{n-1} \otimes e_n v_{(n-1)'} \\ &= (q^{\rho_n} - q^{\rho_{(n-1)'}+1}) v_{n-1} \otimes v_n + (q^{\rho_{n'}+1} - q^{\rho_{n-1}}) v_n \otimes v_{n-1} = 0. \end{aligned}$$

If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, by Lemma 2.1,

$$\begin{aligned} e_n \alpha &= q^{\rho_{n+1}} e_n v_{n+1} \otimes v_{n+1} + q^{\rho_n} e_n v_{n'} \otimes v_n + q^{\rho_{n+1}} k_n v_{n+1} \otimes e_n v_{n+1} \\ &\quad + q^{\rho_{n'}} k_n v_n \otimes e_n v_{n'} \\ &= (q^{\rho_{n+1}} - q^{\rho_{n'}+\frac{1}{2}}) v_n \otimes v_{n+1} + (q^{\rho_{n+1}} - q^{\rho_n-\frac{1}{2}}) v_n \otimes v_{n-1} = 0. \end{aligned}$$

In any case, we have $e_i \alpha = 0$, $1 \leq i \leq n$. Finally, one can check $f_i \alpha = 0$, $1 \leq i \leq n$. \square

²If $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\kappa = \mathbb{Q}(q^{1/2})$, there is a difference between (2) and that in [14, (4.16)], where Hayashi defined $e_n v_k = 0$ unless $k \in \{n+1, n'\}$ and $e_n v_{n+1} = q^{1/2} v_n$, $e_n v_{n'} = -v_{n+1}$ and $f_n v_k = 0$ unless $k \in \{n, n+1\}$ and $f_n v_{n+1} = -v_{n'}$, $f_n v_n = q^{-1/2} v_{n+1}$. In this case, $(e_n f_n - f_n e_n)(v_n) \neq (\frac{k_n - k_n^{-1}}{q^{1/2} - q^{-1/2}})(v_n)$.

Let E_{ij} 's be the matrix units. Consider the operator

$$\begin{aligned} \check{R} = & \sum_{i \neq i'} (qE_{ii} \otimes E_{ii} + q^{-1}E_{ii'} \otimes E_{i'i}) + \sum_{i \neq j, j'} E_{ij} \otimes E_{ji} \\ & + (q - q^{-1}) \sum_{i > j} (E_{jj} \otimes E_{ii} - q^{\rho_i - \rho_j} \varepsilon_i \varepsilon_j E_{ji'} \otimes E_{j'i}) + X, \end{aligned} \quad (2.12)$$

where ε_i 's (resp., ρ) are defined in Corollary 2.3 (resp., (2.11)), and X is $E_{n+1, n+1}^{\otimes 2}$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and 0, otherwise. As in (2.9)–(2.10), we go on identifying $\{1, 2, \dots, 2', 1'\}$ with $\{1, 2, \dots, N\}$. Let $\delta = q - q^{-1}$.

Lemma 2.4. *Let $V = \bigoplus_{i=1}^N \kappa v_i$ be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$.*

(1) *If either $\mathfrak{g} \neq \mathfrak{so}_{2n+1}$ or $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $(k, \ell) \neq (n+1, n+1)$, then*

$$(v_k \otimes v_\ell) \check{R} = \begin{cases} qv_k \otimes v_k, & \text{if } k = \ell, \\ v_\ell \otimes v_k, & \text{if } k > \ell, k \neq \ell', \\ q^{-1}v_\ell \otimes v_k - \delta \sum_{i > k} q^{\rho_i - \rho_k} \varepsilon_i \varepsilon_k v_{i'} \otimes v_i, & \text{if } k > \ell, k = \ell', \\ v_\ell \otimes v_k + \delta v_k \otimes v_\ell, & \text{if } k < \ell, k \neq \ell', \\ q^{-1}v_\ell \otimes v_k + \delta(v_k \otimes v_\ell - \sum_{i > k} q^{\rho_i - \rho_k} \varepsilon_i \varepsilon_k v_{i'} \otimes v_i), & \text{if } k < \ell, k = \ell'. \end{cases} \quad (2.13)$$

(2) *If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then $(v_{n+1} \otimes v_{n+1}) \check{R} = v_{n+1} \otimes v_{n+1} - \delta \sum_{i > n+1} q^{\rho_i} v_{i'} \otimes v_i$,*

(3) *$\check{R} - \check{R}^{-1} = \delta(1 - E)$, where $E : V^{\otimes 2} \rightarrow V^{\otimes 2}$ such that*

$$(v_k \otimes v_\ell) E = \begin{cases} \sum_{i=1}^N q^{\rho_{i'} - \rho_k} \varepsilon_{i'} \varepsilon_k v_i \otimes v_{i'}, & \text{if } k = \ell', \\ 0, & \text{otherwise.} \end{cases} \quad (2.14)$$

Proof. Easy exercise. \square

Following [15], we say that $v_{j_1} \otimes v_{j_2}$ is involved in $(v_{i_1} \otimes v_{i_2}) \check{R}$ if it appears in $(v_{i_1} \otimes v_{i_2}) \check{R}$ with non-zero coefficient. For any positive integers r and N , let

$$I(N, r) = \{(i_1, i_2, \dots, i_r) \mid 1 \leq i_j \leq N, \forall 1 \leq j \leq r\}. \quad (2.15)$$

If $\mathbf{i} \in I(N, r)$, we write

$$v_{\mathbf{i}} = v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r}. \quad (2.16)$$

Corollary 2.5. *Let V be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$. If $(i_1, i_2), (j_1, j_2) \in I(N, 2)$, then $v_{j_1} \otimes v_{j_2} \in V^{\otimes 2}$ is involved in $(v_{i_1} \otimes v_{i_2}) \check{R}$ only if $j_1 \leq i_2$ and $j_2 \geq i_1$.*

Proof. The result was given in [15] for $\mathbf{U}_\kappa(\mathfrak{sp}_{2n})$. The other cases follow from Lemma 2.4, immediately. \square

Definition 2.6. [6, 20] Let R be a commutative ring containing 1 and invertible elements ϱ, q and $q - q^{-1}$. The Birman-Murakami-Wenzl algebra $\mathcal{B}_r(\varrho, q)$ is the unital associative R -algebra generated by $T_i, E_i, 1 \leq i \leq r-1$ satisfying relations

- (1) $(T_i - q)(T_i + q^{-1})(T_i - \varrho^{-1}) = 0$ for $1 \leq i \leq r-1$,
- (2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for $1 \leq i \leq r-2$,
- (3) $T_i T_j = T_j T_i$, for $|i - j| > 1$,
- (4) $E_i T_j^{\pm 1} E_i = \varrho^{\pm 1} E_i$, for $1 \leq i \leq r-1$ and $j = i \pm 1$,
- (5) $E_i T_i = T_i E_i = \varrho^{-1} E_i$, for $1 \leq i \leq r-1$.

where $T_i - T_i^{-1} = (q - q^{-1})(1 - E_i)$ for $1 \leq i \leq r - 1$.

The following results follow from Definition 2.6, immediately.

Lemma 2.7. *Let $\mathcal{B}_r(\varrho, q)$ be defined over R .*

- (1) *There is an R -linear anti-involution σ of $\mathcal{B}_r(\varrho, q)$ fixing T_i and E_i , $1 \leq i \leq r - 1$.*
- (2) *There is an R -linear automorphism γ of $\mathcal{B}_r(\varrho, q)$ such that $\gamma(T_i) = T_{r-i}$ and $\gamma(E_i) = E_{r-i}$, $1 \leq i \leq r - 1$.*
- (3) *Let $\tilde{\sigma} = \sigma \circ \gamma$. Then $\tilde{\sigma}$ is an R -linear anti-involution of $\mathcal{B}_r(\varrho, q)$ such that $\tilde{\sigma}(T_i) = T_{r-i}$ and $\tilde{\sigma}(E_i) = E_{r-i}$, $1 \leq i \leq r - 1$.*

In this paper, we need Enyang's result on a basis of $\mathcal{B}_r(\varrho, q)$ in [12]. Let \mathfrak{S}_r be the symmetric group in r letters $\{1, 2, \dots, r\}$. Then \mathfrak{S}_r is a Coxeter group with generators s_1, s_2, \dots, s_{r-1} satisfying usual braid relations together with $s_i^2 = 1$, $1 \leq i \leq r - 1$. For each integer f , $1 \leq f \leq \lfloor \frac{r}{2} \rfloor$, let \mathfrak{B}_f be the subgroup of \mathfrak{S}_r generated by s_1 , and $s_{2i-2}s_{2i-1}s_{2i-3}s_{2i-2}$, $2 \leq i \leq f$. If $f = 0$, we set $\mathfrak{B}_f = 1$. Enyang [12] described \mathcal{D}_f , a complete set of right coset representatives of $\mathfrak{B}_f \times \mathfrak{S}_{r-2f}$ in \mathfrak{S}_r , where \mathfrak{S}_{r-2f} is the subgroup of \mathfrak{S}_r generated by s_j , $2f + 1 \leq j \leq r - 1$. For any $w \in \mathfrak{S}_r$, write $T_w = T_{i_1}T_{i_2} \cdots T_{i_k} \in \mathcal{B}_r(\varrho, q)$ if $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w . It is known that T_w is independent of a reduced expression of w .

Theorem 2.8. [12] *Suppose that R is a commutative ring containing 1 and invertible elements ϱ, q and $q - q^{-1}$. Then $S_1 = \{T_{d_1}^* E^f T_w T_{d_2} \mid 0 \leq f \leq \lfloor r/2 \rfloor, w \in \mathfrak{S}_{r-2f}, d_1, d_2 \in \mathcal{D}_f\}$ is an R -basis of $\mathcal{B}_r(\varrho, q)$, where $E^f = E_1 E_3 \cdots E_{2f-1}$ for $f > 0$ and $E^0 = 1$, and $*$ is the R -linear anti-involution σ on $\mathcal{B}_r(\varrho, q)$ given in Lemma 2.7(1).*

Let \mathcal{D}^f be the set of distinguished right coset representatives of \mathfrak{B}_f in the subgroup \mathfrak{S}_{2f} of \mathfrak{S}_r generated by s_i , $1 \leq i \leq 2f - 1$. It was defined in [9] that

$$P_f = \{(i_1, i_2, \dots, i_{2f}) \mid 1 \leq i_1 < \dots < i_{2f} \leq r\}. \quad (2.17)$$

For each $J \in P_f$, define

$$d_J = s_{2f, i_{2f}} s_{2f-1, i_{2f-1}} \cdots s_{2, i_2} s_{1, i_1}, \quad (2.18)$$

where $s_{i,j} = s_i s_{i+1, j}$ (resp., 1) for $i < j$ (resp., $i = j$) and $s_{i,j} = s_{j,i}^{-1}$ if $i > j$. Then d_J is the unique element in \mathcal{D}_f such that $(k)d_J = i_k$, $1 \leq k \leq 2f$. Further, by [9, Lemma 3.8],

$$\mathcal{D}_f = \dot{\bigcup}_{J \in P_f} \mathcal{D}^f d_J, \quad (2.19)$$

where $\dot{\bigcup}$ denotes a disjoint union. Following [15], define $J_0 = (r - 2f + 1, \dots, r - 1, r) \in P_f$ and $d_0 = s_{2f-2, 2f} s_{2f-4, 2f} \cdots s_{2, 2f} \in \mathcal{D}^f$.

Lemma 2.9. [15, Lemma 5.12]

- (1) *For any $d \in \mathcal{D}_f$, there is a $w \in \mathfrak{S}_r$, such that $d_0 = dw$ and $\ell(d_0) = \ell(d) + \ell(w)$, where $\ell(\cdot)$ is the length function on \mathfrak{S}_r .*
- (2) *For any $J \in P_f$, there is a $w' \in \mathfrak{S}_r$, such that $d_{J_0} = d_J w'$ and $\ell(d_{J_0}) = \ell(d_J) + \ell(w')$.*
- (3) *For any $d \in \mathcal{D}_f$ with $d \neq d_0 d_{J_0}$, there is a j with $1 \leq j < r$, such that $ds_j \in \mathcal{D}_f$ and $\ell(ds_j) = \ell(d) + 1$.*

In the remaining part of this section, we always assume that

$$\varrho = \begin{cases} -q^{2n+1}, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ q^{2n-1}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}, \\ q^{2n}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1}. \end{cases} \quad (2.20)$$

Let V be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$ with $\mathfrak{g} \in \{\mathfrak{sp}_{2n}, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$. If ϱ is given in (2.20), then there is a κ -algebra homomorphism

$$\varphi : \mathcal{B}_r(\varrho, q) \rightarrow \text{End}_{\mathbf{U}_\kappa(\mathfrak{g})}(V^{\otimes r}) \quad (2.21)$$

such that

$$\varphi(T_i) = 1^{\otimes i-1} \otimes \check{R} \otimes 1 \otimes \cdots \otimes 1 \text{ and } \varphi(E_i) = 1^{\otimes i-1} \otimes E \otimes 1 \otimes \cdots \otimes 1. \quad (2.22)$$

We remark that φ has been defined in [14] when κ is $\mathbb{C}(v)$. However, since V contains an \mathcal{A} -lattice which is a left $\mathbf{U}(\mathfrak{g})$ -module, by base change, it can be defined over an arbitrary field κ .

In the remaining part of this section, all results for $\mathbf{U}_\kappa(\mathfrak{sp}_{2n})$ have been proved in [15]. The corresponding results for both $\mathbf{U}_\kappa(\mathfrak{so}_{2n})$ and $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$ can also be proved by arguments in [15]. For self-contained reason, we give a sketch.

Lemma 2.10. (cf. [15, Lemma 5.6]) *Suppose $n \geq r$. Then $\ker \varphi \subseteq \mathcal{B}_r(\varrho, q)^1$, where $\mathcal{B}_r(\varrho, q)^f$ is the two-sided ideal of $\mathcal{B}_r(\varrho, q)$ generated by E^f , $1 \leq f \leq [r/2]$.*

Proof. Recall that $\{v_i \mid 1 \leq i \leq N\}$ is a basis of V . Let $v = v_r \otimes v_{r-1} \otimes \cdots \otimes v_1 \in V^{\otimes r}$. If $x \in \ker \varphi$, then $vx = 0$. It is proved in [15, Lemma 5.6] that $x \in \mathcal{B}_r(\varrho, q)^1$ if $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $n \geq r$. By (2.13), $\mathcal{B}_r(\varrho, q)$ acts on v via the same formula for $\mathfrak{g} \in \{\mathfrak{sp}_{2n}, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$. So, the results for $\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}$ follow from similar arguments. \square

For $\mathbf{i} \in I(N, r)$, let $\ell(v_{\mathbf{i}}) = \ell(\mathbf{i})$, which is the maximal number of disjoint pairs (s, t) such that $i_s = (i_t)'$. When $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\ell(v_{\mathbf{i}})$ is called the *symplectic length* of \mathbf{i} in [15]. The following result has been given in [15, Lemma 5.14] for $\mathfrak{g} = \mathfrak{sp}_{2n}$. In Cases 2–3 of the proof of [15, Lemma 5.14], Hu used [15, (5.13)] and did not use the explicit description of $(v_{i_1} \otimes v_{i_2})\check{R}$. If $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$, [15, (5.13)] is still true (see Corollary 2.5). So, arguments in the proof of [15, Lemma 5.14] can be used smoothly to give the proof of the corresponding results for both \mathfrak{so}_{2n} and \mathfrak{so}_{2n+1} as follows³.

Lemma 2.11. (cf. [15, Lemma 5.14]) *Fix a positive integer s with $1 \leq s \leq f$ and assume that $\mathbf{i} \in I(N, a)$ such that either $1 \leq i_j \leq n - f$ or $n' \leq i_j \leq (n - f + s + 1)'$ for each integer j with $1 \leq j \leq a$. Suppose that d is a distinguish right coset representative of $\mathfrak{S}_{2s, a}$ in \mathfrak{S}_{2s+a} , where $\mathfrak{S}_{2s, a}$ is the subgroup of \mathfrak{S}_{2s+a} generated by s_j with $j \neq 2s$. If $J = (a + 1, a + 2, \dots, a + 2s)$, and $\mathbf{j} = ((n - f + s)', \dots, (n - f + 2)', (n - f + 1)', n - f + 1, \dots, n - f + s)$, then*

$$(v_{\mathbf{i}} \otimes v_{\mathbf{j}})T_{d^{-1}} = q^z \delta_{d, d_J} v_{\mathbf{j}} \otimes v_{\mathbf{i}} + \sum_{\mathbf{u} \in I(N, 2s+a)} a_{\mathbf{u}} v_{\mathbf{u}},$$

for some $z \in \mathbb{Z}$ such that $a_{\mathbf{u}} \neq 0$ only if $\ell(u_1, \dots, u_{2s}) < s$, and $x \notin \{u_1, u_2, \dots, u_{2s}\}$ for any positive integer x satisfying either $(n - f)' \leq x \leq 1'$ or $n - f + s + 1 \leq x \leq n$.

³ We remark that \tilde{w} in [15, Lemma 5.14] should be read as $v_{\mathbf{j}}$ in Lemma 2.11 so that one can get a suitable induction assumption in Cases 2–3 in the proof of [15, Lemma 5.14].

Following [15], let

$$I_f = \{(b_1, \dots, b_{r-2f}) \mid 1 \leq b_{r-2f} < \dots < b_2 < b_1 \leq n - f\}. \quad (2.23)$$

The following result is the counterpart of [15, Lemma 5.15]. It can be proved by arguments similar to those in the proof of [15, Lemma 5.15]. The difference is that one needs to use Lemma 2.11 instead of [15, Lemma 5.14].

Lemma 2.12. (cf. [15, Lemma 5.15]) Suppose $v = v_{\mathbf{b}} \otimes v_{\mathbf{c}} \in V^{\otimes r}$ for some $\mathbf{b} \in I_f$ and $\mathbf{c} = (n', (n-1)', \dots, (n-f+1)', n-f+1, \dots, n-1, n)^4$. If $w \in \mathcal{D}_f$ such that $w \neq d_0 d_{J_0}$, then $(v)T_w^* E^f = 0$.

For any $v \in V^{\otimes r}$, let $\text{ann}(v) = \{x \in \mathcal{B}_r(\varrho, q) \mid vx = 0\}$. The following result, which is the key step in the proof of the injectivity of φ , is the counterpart of [15, Lemma 5.18].

Lemma 2.13. (cf. [15, Lemma 5.18]) Let M be the κ -space spanned by

$$S = \{T_{d_1}^* E^f T_{d_2} \mid d_1, d_2 \in \mathcal{D}_f, d_1 \neq d_0 d_{J_0}, \sigma \in \mathfrak{S}_{r-2f}\}.$$

Then $\mathcal{B}_r(\varrho, q)^f \cap \bigcap_{\mathbf{b} \in I_f} \text{ann}(v_{\mathbf{b}} \otimes v_{\mathbf{c}}) = \mathcal{B}_r(\varrho, q)^{f+1} \oplus M$.

Proof. Note that $\ell(v_{\mathbf{b}}) = 0$ for any $\mathbf{b} \in I_f$. So, $\mathcal{B}_r(\varrho, q)^{f+1} \subseteq \text{ann}(v_{\mathbf{b}} \otimes v_{\mathbf{c}})$. By Lemma 2.12, we have the result for " \supseteq ". Conversely, for any $x \in \mathcal{B}_r(\varrho, q)^f \cap \bigcap_{\mathbf{b} \in I_f} \text{ann}(v_{\mathbf{b}} \otimes v_{\mathbf{c}})$, By Theorem 2.8 and Lemma 2.12, we can write

$$x = T_{d_0 d_{J_0}}^* E^f \left(\sum_{d \in \mathcal{D}_f} z_d T_d \right) + h,$$

where $z_d = \sum_{\sigma \in \mathfrak{S}_{r-2f}} a_{\sigma} T_{\sigma}$, for $d \in \mathcal{D}_f$, $a_{\sigma} \in \kappa$ and $h \in \mathcal{B}_r(\varrho, q)^{f+1} \oplus M$. In order to prove the result for " \subseteq ", it suffices to show $z_d = 0$, for each $d \in \mathcal{D}_f$. In [15], Hu proved $z_{d_0 d_{J_0}} = 0$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$. Further, since his proof depends on [15, (5.13)] and does not depend on the explicit description of $(v_{i_1} \otimes v_{i_2})\tilde{R}$, one can use Corollary 2.5 to replace [15, (5.13)] in Step 1 in the proof of [15, Lemma 5.18]. So, $z_{d_0 d_{J_0}} = 0$. By Lemma 2.9, $d_0 d_{J_0}$ is the maximal element of \mathcal{D}_f with respect to the Bruhat order. Mimicking arguments in the proof of Step 2 of [15, Lemma 5.18], i.e. by induction on $\ell(d)$ for $d \in \mathcal{D}_f$, one can verify $z_d = 0$ for $d \neq d_0 d_{J_0}$. \square

The following result, which is [15, Theorem 5.19] for $\mathfrak{g} = \mathfrak{sp}_{2n}$, can be verified via arguments on induction of $\ell(d)$ in the proof of [15, Theorem 5.19].

Lemma 2.14. (cf. [15, Theorem 5.19]) $\ker \varphi \subseteq \mathcal{B}_r(\varrho, q)^{f+1}$ if $\ker \varphi \subseteq \mathcal{B}_r(\varrho, q)^f$.

Theorem 2.15. Let V be the natural representation of $\mathbf{U}_{\kappa}(\mathfrak{g})$ with $\mathfrak{g} \in \{\mathfrak{sp}_{2n}, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$. Then φ defined in (2.21) is a κ -algebra isomorphism if

- (1) $\mathfrak{g} = \mathfrak{sp}_{2n}$ with $n \geq r$,
- (2) $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$ with $n > r$.

Proof. We remark that (1) has been proved in [15]. If $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}\}$, φ is well-defined over κ (in fact, over R). Further, by Lemma 2.10 and Lemma 2.14, $\ker \varphi \in \mathcal{B}_r(\varrho, q)^f$ for all positive integers f , forcing $\ker \varphi = 0$. In order to complete proof, it is enough to show that the dimensions of $\mathcal{B}_r(\varrho, q)$ and $\text{End}_{\mathbf{U}_{\kappa}(\mathfrak{g})}(V^{\otimes r})$ are the same. It was defined in [1, Definition 2.1]

⁴The element $v_{\mathbf{c}}$ in [15, Lemma 5.18] should be read as current $v_{\mathbf{c}}$ so as to be compatible with $v_{\mathbf{j}}$ in Lemma 2.11.

that a tilting module for $\mathbf{U}_\kappa(\mathfrak{g})$ is a finite dimensional left $\mathbf{U}_\kappa(\mathfrak{g})$ -module which has a Weyl-filtration and a co-Weyl filtration. Since $V = \Delta(\epsilon_1)$, the Weyl module with highest weight ϵ_1 , and $V \cong V^*$, V is a tilting module for $\mathbf{U}_\kappa(\mathfrak{g})$ and so is $V^{\otimes r}$. By Lemma 5.1 in [3], the dimension of $\text{End}_{\mathbf{U}_\kappa(\mathfrak{g})}(V^{\otimes r})$ is independent of κ . In particular, we assume $\kappa = \mathbb{C}(v)$ where v is an indeterminate. In this case, $V^{\otimes r}$ is completely reducible. By [18, (5.5)] and Enyang's construction of Jucys-Murphy basis of $\mathcal{B}_r(\varrho, q)$ in [12], the dimension of $\text{End}_{\mathbf{U}_\kappa(\mathfrak{g})}(V^{\otimes r})$ is equal to that of $\mathcal{B}_r(\varrho, q)$. So, φ is surjective. \square

3. AN INVARIANT FORM ON $V^{\otimes r}$

In this section, we always assume that κ is a field containing q (resp., $q^{1/2}$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$) such that $q^2 \neq 1$. Let V be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$, with $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}\}$. The aim of this section is to prove that $V^{\otimes r}$ is self-dual as $(\mathbf{U}_\kappa(\mathfrak{g}), \mathcal{B}_r(\varrho, q))$ -module if ϱ is given in (2.20).

First, we consider $\mathfrak{g} = \mathfrak{so}_{2n+1}$. For any $\mathbf{i} \in I(2n+1, r)$, define $\mathbf{i}' \in I(2n+1, r)$ such that $\mathbf{i}' = (i'_r, i'_{r-1}, \dots, i'_1)$ if $\mathbf{i} = (i_1, \dots, i_r)$, where $i' = 2n+2-i$, and $i'' = i$, $1 \leq i \leq n$.

Lemma 3.1. *For any positive integer r , define the κ -bilinear form $\langle \cdot, \cdot \rangle : V^{\otimes r} \times V^{\otimes r} \rightarrow \kappa$ such that*

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = q^{-\rho_{\mathbf{i}}} \delta_{\mathbf{i}, \mathbf{j}'}, \quad \text{for } \mathbf{i}, \mathbf{j} \in I(2n+1, r), \quad (3.1)$$

where $\rho_{\mathbf{i}} = \sum_{k=1}^r \rho_{i_k}$, and ρ is given in (2.11).

- (1) *The bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate.*
- (2) *$\langle av_{\mathbf{i}}, v_{\mathbf{j}} \rangle = \langle v_{\mathbf{i}}, S(a)v_{\mathbf{j}} \rangle$, $a \in \mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$, $\mathbf{i}, \mathbf{j} \in I(2n+1, r)$, where S is the antipode of $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$ given in (2.4).*
- (3) *$\langle v_{\mathbf{i}}b, v_{\mathbf{j}} \rangle = \langle v_{\mathbf{i}}, v_{\mathbf{j}}\tilde{\sigma}(b) \rangle$, $b \in \mathcal{B}_r(\varrho, q)$, $\mathbf{i}, \mathbf{j} \in I(2n+1, r)$, where $\tilde{\sigma}$ is the anti-involution on $\mathcal{B}_r(\varrho, q)$ given in Lemma 2.7(3).*

Proof. We remark that (1) follows from (3.1), immediately. Let V^* be the κ -linear dual of V . Then $V \cong V^*$ as left $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$ -modules and the corresponding isomorphism φ satisfies

$$\varphi(v_i) = q^{-\rho_i} v_{i'}^*, \quad 1 \leq i \leq 2n+1, \quad (3.2)$$

where $\{v_i^* \mid 1 \leq i \leq 2n+1\}$ is the dual basis of a basis $\{v_i \mid 1 \leq i \leq 2n+1\}$ of V . By Proposition 111.5.2 in [8], $M^* \otimes N^* \cong (N \otimes M)^*$ for any finite dimensional $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$ -modules M and N . So $(V^*)^{\otimes r} \cong (V^{\otimes r})^*$ and the corresponding isomorphism $\Psi : (V^*)^{\otimes r} \rightarrow (V^{\otimes r})^*$ satisfies

$$\Psi(v_{i_1}^* \otimes \dots \otimes v_{i_r}^*) = (v_{i_r} \otimes \dots \otimes v_{i_1})^*, \quad \mathbf{i} \in I(2n+1, r). \quad (3.3)$$

Thus

$$\Phi : V^{\otimes r} \cong (V^{\otimes r})^* \quad (3.4)$$

as left $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$ -modules where $\Phi = \Psi \circ \varphi^{\otimes r}$. It is routine to check that

$$\Phi(v_{\mathbf{i}})(v_{\mathbf{j}}) = \langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle, \quad \forall \mathbf{i}, \mathbf{j} \in I(2n+1, r). \quad (3.5)$$

Now, (2) follows since it is equivalent to saying that Φ is a left $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$ -homomorphism. By Definition 2.6, $\mathcal{B}_r(\varrho, q)$ can be generated by $T_i^{\pm 1}$, $1 \leq i \leq r-1$. In order to prove (3), by (3.1), it suffices to verify

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_1 \rangle = \langle v_{\mathbf{i}} T_1, v_{\mathbf{j}} \rangle \quad (3.6)$$

for $r = 2$. If so, we have $\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_1^{-1} \rangle = \langle v_{\mathbf{i}} T_1^{-1} T_1, v_{\mathbf{j}} T_1^{-1} \rangle = \langle v_{\mathbf{i}} T_1^{-1}, v_{\mathbf{j}} \rangle$, proving (3).

By (2.13), it is easy to check (3.6) if $i_1 \neq i'_2$. Assume $i_1 = i'_2$ and write $\delta = q - q^{-1}$. If $\mathbf{i} = (i_1, i_2) = (n+1, n+1)$, then

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_1 \rangle = \langle v_{\mathbf{i}} T_1, v_{\mathbf{j}} \rangle = \begin{cases} 1, & j_1 = j_2 = n+1, \\ -\delta q^{-\rho_{j_1}}, & j_2 = j'_1 > n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\mathbf{i} \neq (n+1, n+1)$. If $i_1 > i_2$, then

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_1 \rangle = \langle v_{\mathbf{i}} T_1, v_{\mathbf{j}} \rangle = \begin{cases} q^{-1}, & (j_1, j_2) = (i_2, i_1), \\ -\delta q^{\rho_{j_2} - \rho_{i_1}}, & j_2 = j'_1 > i_1, \\ 0, & \text{otherwise.} \end{cases}$$

If $i_1 < i_2$ and $\mathbf{j} = \mathbf{i}$, then (3.6) follows from (3.1). If $i_1 < i_2$ and $\mathbf{j} \neq \mathbf{i}$,

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_1 \rangle = \langle v_{\mathbf{i}} T_1, v_{\mathbf{j}} \rangle = \begin{cases} q^{-1}, & (j_1, j_2) = (i_2, i_1), \\ -\delta q^{\rho_{j_2} - \rho_{i_1}}, & i_2 \neq j_2 = j'_1 > i_1, \\ 0, & \text{otherwise.} \end{cases}$$

In any case, we have (3.6), proving (3). \square

For any right $\mathcal{B}_r(\varrho, q)$ -module M , M^* is a right $\mathcal{B}_r(\varrho, q)$ -module such that

$$(\phi b)(x) = \phi(x \tilde{\sigma}(b)), \forall \phi \in M^*, b \in \mathcal{B}_r(\varrho, q), x \in M, \quad (3.7)$$

where $\tilde{\sigma}$ is the anti-involution on $\mathcal{B}_r(\varrho, q)$ given in Lemma 2.7(3).

Corollary 3.2. *As $(\mathbf{U}_{\kappa}(\mathfrak{so}_{2n+1}), \mathcal{B}_r(\varrho, q))$ -bimodules, $V^{\otimes r} \cong (V^{\otimes r})^*$ where ϱ is given in (2.20).*

Proof. By Lemma 3.1(2)–(3) and (3.5), the Φ given in (3.4) is the required isomorphism. \square

Now, we assume $\mathfrak{g} \in \{\mathfrak{sp}_{2n}, \mathfrak{so}_{2n}\}$. Recall that $\tau : \mathbf{U}_{\kappa}(\mathfrak{g}) \rightarrow \mathbf{U}_{\kappa}(\mathfrak{g})$ is an anti-automorphism such that

$$\tau(k_i) = k_i, \tau(e_i) = f_i \text{ and } \tau(f_i) = e_i, \quad 1 \leq i \leq n. \quad (3.8)$$

For any left $\mathbf{U}_{\kappa}(\mathfrak{g})$ -module N , let N° be the left $\mathbf{U}_{\kappa}(\mathfrak{g})$ -module such that $N^{\circ} = N^*$ as κ -vector spaces, and the action is given by

$$(u\phi)(x) = \phi(\tau(u)x), \forall x \in N, u \in \mathbf{U}_{\kappa}(\mathfrak{g}), \phi \in N^*. \quad (3.9)$$

Let $\varrho \in \kappa$ be given in (2.20). For any right $\mathcal{B}_r(\varrho, q)$ -module M , let M° be the right $\mathcal{B}_r(\varrho, q)$ -module such that $M^{\circ} = M^*$ as κ -vector spaces, and the action is given by

$$(\phi b)(y) = \phi(y\sigma(b)), \forall y \in M, b \in \mathcal{B}_r(\varrho, q), \phi \in M^*, \quad (3.10)$$

where σ is the anti-involution on $\mathcal{B}_r(\varrho, q)$ given in Lemma 2.7.

Lemma 3.3. *For any positive integer r , let $\langle \cdot, \cdot \rangle : V^{\otimes r} \times V^{\otimes r} \rightarrow \kappa$ be the bilinear form such that*

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = q^{\beta(\mathbf{i})} \delta_{\mathbf{i}, \mathbf{j}}, \quad \forall \mathbf{i}, \mathbf{j} \in I(2n, r), \quad (3.11)$$

where $\beta(\mathbf{i}) = \#\{i_k \neq i_j \mid j \neq k, i_j \neq i'_k\} + 2\#\{i_k = i'_j\}$. Then

- (1) $\langle \cdot, \cdot \rangle$ is symmetric and non-degenerate.
- (2) $\langle uv, w \rangle = \langle v, \tau(u)w \rangle$, $\forall u \in \mathbf{U}_{\kappa}(\mathfrak{g})$ and $v, w \in V^{\otimes r}$, where τ is the anti-automorphism of $\mathbf{U}_{\kappa}(\mathfrak{g})$ given in (3.8).

- (3) $\langle vb, w \rangle = \langle v, w\sigma(b) \rangle$, $\forall b \in \mathcal{B}_r(\varrho, q)$ and $v, w \in V^{\otimes r}$, where σ is the anti-involution defined in Lemma 2.7.

Proof. (1) follows from (3.11), immediately. In order to prove (2), it suffices to verify

$$\langle uv, w \rangle = \langle v, \tau(u)w \rangle \quad (3.12)$$

for all $v, w \in V^{\otimes r}$ and $u \in \{e_i, f_i, k_i \mid 1 \leq i \leq n\}$. It is easy to check (3.12) if $u = k_i$. Since $\langle \cdot, \cdot \rangle$ is symmetric, it remains to check (3.12) when $u = e_i$, $1 \leq i \leq n$.

First, we assume $i \neq n$. Suppose $v = v_{\mathbf{i}}$ and $w = v_{\mathbf{j}}$ for $\mathbf{i}, \mathbf{j} \in I(2n, r)$. Then $\langle e_i v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = 0$ unless there is a k , $1 \leq k \leq r$ such that $(i_k, j_k) \in \{(i+1, i), (i', (i+1)')\}$ and $j_l = i_l$ for all $l \neq k$. In the later case, let α_a (resp., γ_a) be the numbers of a appearing in (i_1, \dots, i_{k-1}) (resp., (i_{k+1}, \dots, i_r)). Then

$$\beta(\mathbf{i}) = \beta(\mathbf{j}) + \alpha_i + \alpha_{(i+1)'} - \alpha_{i'} - \alpha_{i+1} + \gamma_i + \gamma_{(i+1)'} - \gamma_{i'} - \gamma_{i+1}. \quad (3.13)$$

It is routine to check

$$\langle e_i v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = (-1)^{\delta_{i_k, i'}} q^{\alpha_i + \alpha_{(i+1)'} - \alpha_{i'} - \alpha_{i+1} + \beta(\mathbf{j})}, \quad (3.14)$$

and

$$\langle v_{\mathbf{i}}, f_i v_{\mathbf{j}} \rangle = (-1)^{\delta_{i_k, i'}} q^{-\gamma_i - \gamma_{(i+1)'} + \gamma_{i'} + \gamma_{i+1} + \beta(\mathbf{i})}. \quad (3.15)$$

By (3.13)–(3.15), $\langle e_i v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = \langle v_{\mathbf{i}}, f_i v_{\mathbf{j}} \rangle$ if $\langle e_i v_{\mathbf{i}}, v_{\mathbf{j}} \rangle \neq 0$. Finally, it is easy to check $\langle e_i v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = 0$ if and only if $\langle v_{\mathbf{i}}, f_i v_{\mathbf{j}} \rangle = 0$.

Suppose $i = n$. We have $\langle e_n v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = 0$ unless one of two conditions holds:

- (a) $j_l = i_l$ unless $j = k$ for some k , $1 \leq k \leq r$ and $(i_k, j_k) = (n', n)$ provided $\mathfrak{g} = \mathfrak{sp}_{2n}$;
- (b) $j_l = i_l$ unless $l = k$ for some k , $1 \leq k \leq r$ and $(i_k, j_k) \in \{(n', n-1), ((n-1)', n)\}$ provided $\mathfrak{g} = \mathfrak{so}_{2n}$.

In case (a), $\beta(\mathbf{i}) = \beta(\mathbf{j}) + 2\alpha_n - 2\alpha_{n'} + 2\gamma_n - 2\gamma_{n'}$, and hence

$$\langle e_n v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = q^{2\alpha_n - 2\alpha_{n'} + \beta(\mathbf{j})} = q^{-2\gamma_n + 2\gamma_{n'} + \beta(\mathbf{i})} = \langle v_{\mathbf{i}}, f_n v_{\mathbf{j}} \rangle.$$

In case (b), $\beta(\mathbf{i}) - \gamma_n - \gamma_{n-1} + \gamma_{n'} + \gamma_{(n-1)'} = \beta(\mathbf{j}) + \alpha_n + \alpha_{n-1} - \alpha_{n'} - \alpha_{(n-1)'} = a$, and

$$\langle e_n v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = (-1)^\epsilon q^a = \langle v_{\mathbf{i}}, f_n v_{\mathbf{j}} \rangle,$$

where $\epsilon = 0$ (resp., 1) if $i_k = n'$ (resp., $i_k = (n-1)'$). In any case, we have (3.12) if $\langle e_n v_{\mathbf{i}}, v_{\mathbf{j}} \rangle \neq 0$. Finally, it is easy to see that $\langle e_n v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = 0$ if and only if $\langle v_{\mathbf{i}}, f_n v_{\mathbf{j}} \rangle = 0$. This completes the proof of (2).

In order to verify (3), it suffices to assume $v = v_{\mathbf{i}}$, $w = v_{\mathbf{j}}$ and $b = T_k$, $\forall \mathbf{i}, \mathbf{j} \in I(2n, r)$ and $1 \leq k \leq r-1$. We assume $i_l = j_l$, for $l \neq k, k+1$. Otherwise, $\langle v_{\mathbf{i}} T_k, v_{\mathbf{j}} \rangle = \langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle = 0$. By Lemma 2.4, $\langle v_{\mathbf{i}} T_k, v_{\mathbf{j}} \rangle = \langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle$ if $i_k \neq i'_{k+1}$. In the remaining, we assume $i_k = i'_{k+1}$. In particular, $i_k \neq i_{k+1}$. Write $\delta = q - q^{-1}$. If $i_k > i_{k+1}$, then

$$\langle v_{\mathbf{i}} T_k, v_{\mathbf{j}} \rangle = \begin{cases} q^{-1+\beta(\mathbf{j})}, & (j_k, j_{k+1}) = (i_{k+1}, i_k), \\ -\delta q^{\rho_{j_{k+1}} - \rho_{i_k} + \beta(\mathbf{j})} \varepsilon_{j_{k+1}} \varepsilon_{i_k}, & j_{k+1} = j'_k > i_k, \\ 0, & \text{otherwise.} \end{cases} \quad (3.16)$$

and

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle = \begin{cases} q^{-1+\beta(\mathbf{i})}, & (j_k, j_{k+1}) = (i_{k+1}, i_k), \\ -\delta q^{\rho_{i'_k} - \rho_{j'_{k+1}} + \beta(\mathbf{i})} \varepsilon_{j'_{k+1}} \varepsilon_{i'_k}, & j_{k+1} = j'_k > i_k, \\ 0, & \text{otherwise.} \end{cases} \quad (3.17)$$

If $i_k < i_{k+1}$, by (1), we can assume $\mathbf{i} \neq \mathbf{j}$ without loss of generality. We have

$$\langle v_{\mathbf{i}} T_k, v_{\mathbf{j}} \rangle = \begin{cases} q^{-1+\beta(\mathbf{j})}, & (j_k, j_{k+1}) = (i_{k+1}, i_k), \\ -\delta q^{\rho_{j_{k+1}} - \rho_{i_k} + \beta(\mathbf{j})} \varepsilon_{j_{k+1}} \varepsilon_{i_k}, & i'_{k+1} \neq j_{k+1} = j'_k > i_k, \\ 0, & \text{otherwise.} \end{cases} \quad (3.18)$$

and

$$\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle = \begin{cases} q^{-1+\beta(\mathbf{i})}, & (j_k, j_{k+1}) = (i_{k+1}, i_k), \\ -\delta q^{\rho_{i'_k} - \rho_{j'_{k+1}} + \beta(\mathbf{i})} \varepsilon_{j'_{k+1}} \varepsilon_{i'_k}, & i'_{k+1} \neq j_{k+1} = j'_k > i_k, \\ 0, & \text{otherwise.} \end{cases} \quad (3.19)$$

So, $\langle v_{\mathbf{i}} T_k, v_{\mathbf{j}} \rangle = 0$ if and only if $\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle = 0$. Further, if $\langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle \neq 0$, then $\beta(\mathbf{i}) = \beta(\mathbf{j})$ and hence $\langle v_{\mathbf{i}} T_k, v_{\mathbf{j}} \rangle = \langle v_{\mathbf{i}}, v_{\mathbf{j}} T_k \rangle$ by (3.16)–(3.19), (2.11) and the definition of ε_i in Corollary 2.3.

□

Corollary 3.4. *Suppose $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{sp}_{2n}\}$. As $(\mathbf{U}_{\kappa}(\mathfrak{g}), \mathcal{B}_r(\varrho, q))$ -bimodules, $V^{\otimes r} \cong (V^{\otimes r})^{\circ}$, where ϱ is given in (2.20).*

Proof. Let $\circ : V^{\otimes r} \rightarrow (V^{\otimes r})^{\circ}$ be κ -linear map such that

$$x^{\circ}(y) = \langle x, y \rangle, \quad \forall x, y \in V^{\otimes r}, \quad (3.20)$$

where $\langle \cdot, \cdot \rangle$ is given in (3.11). By Lemma 3.3, \circ is the required isomorphism. □

4. REPRESENTATIONS OF BIRMAN-MURAKAMI-WENZL ALGEBRAS

In this section, we assume that $\mathcal{B}_r(\varrho, q)$ is defined over κ , where κ is a field containing non-zero ϱ and q such that $q^2 \neq 1$. The aim of this section is to establish a relationship between decomposition numbers of $\mathcal{B}_r(\varrho, q)$ and the multiplicities of Weyl modules in certain indecomposable tilting modules for $\mathbf{U}_{\kappa}(\mathfrak{g})$ over κ , where ϱ is given in (2.20) and $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}\}$. We start by recalling some of combinatorics.

Recall that a composition λ of r with at most n parts is a sequence of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\sum_{i=1}^n \lambda_i = r$. If $\lambda_i \geq \lambda_{i+1}$ for all possible i 's, then λ is called a partition. Let λ' be the conjugate of λ . Then $\lambda'_k = \#\{j \mid \lambda_j \geq k\}$. Let $\Lambda(n, r)$ (resp., $\Lambda^+(n, r)$) be the set of all compositions (resp., partitions) of r with at most n parts. We also use $\Lambda^+(r)$ to denote the set of all partitions of r . For any $\lambda \in \Lambda^+(r)$, let $[\lambda]$ be the Young diagram which is a collection of boxes (or nodes) arranged in left-justified rows with λ_i boxes in the i th row of $[\lambda]$. A λ -tableau \mathfrak{s} is obtained by inserting $i, 1 \leq i \leq r$ into $[\lambda]$ without repetition. A λ -tableau \mathfrak{s} is standard if the entries in \mathfrak{s} are increasing both from left to right in each row and from top to the bottom in each column. Let $\mathcal{T}^s(\lambda)$ be the set of all standard λ -tableaux. The symmetric group \mathfrak{S}_r acts on \mathfrak{s} by permuting its entries. Let \mathfrak{t}^{λ} (resp., \mathfrak{t}_{λ}) be the λ -tableau obtained from the Young diagram $[\lambda]$ by adding $1, 2, \dots, n$ from left to right along the rows (resp., from top to bottom down the columns). For example, if $\lambda = (4, 3, 1)$, then

$$\mathfrak{t}^{\lambda} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & & & \\ \hline \end{array}, \quad \text{and } \mathfrak{t}_{\lambda} = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 & \\ \hline 3 & & & \\ \hline \end{array}. \quad (4.1)$$

Write $w = d(\mathfrak{s})$ if $\mathfrak{t}^{\lambda} w = \mathfrak{s}$. Then $d(\mathfrak{s})$ is uniquely determined by \mathfrak{s} . In particular, we denote $d(\mathfrak{t}_{\lambda})$ by w_{λ} .

Let \mathcal{H}_r be the Hecke algebra associated to the symmetric group \mathfrak{S}_r . By definition, \mathcal{H}_r is a unital associative $\mathbb{Z}[q, q^{-1}]$ -algebra generated by $g_i, 1 \leq i \leq r-1$ satisfying relations

- (1) $(g_i - q)(g_i + q^{-1}) = 0, 1 \leq i \leq r-1,$
- (2) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, 1 \leq i \leq r-2,$
- (3) $g_i g_j = g_j g_i, |i - j| > 1.$

Let I be the two-sided ideal of $\mathcal{B}_r(\varrho, q)$ generated by E_1 . By Definition 2.6,

$$\mathcal{H}_r \cong \mathcal{B}_r(\varrho, q)/I. \quad (4.2)$$

For any $w \in \mathfrak{S}_r$, write $g_w = g_{i_1} g_{i_2} \cdots g_{i_k}$ if $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w . It is known that k , the length of w , is unique although a reduced expression of w is not unique in general. For each partition λ of r , let

$$\mathbf{m}_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{\ell(w)} g_w, \text{ and } \mathbf{n}_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-\ell(w)} g_w, \quad (4.3)$$

where \mathfrak{S}_λ is the Young subgroup of \mathfrak{S}_r with respect to λ . For any $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)$, let

$$\mathbf{m}_{\mathfrak{s}\mathfrak{t}} = g_{d(\mathfrak{s})}^* \mathbf{m}_\lambda g_{d(\mathfrak{t})}, \quad \mathbf{n}_{\mathfrak{s}\mathfrak{t}} = g_{d(\mathfrak{s})}^* \mathbf{n}_\lambda g_{d(\mathfrak{t})},$$

where $*$ is the anti-involution on \mathcal{H}_r such that $g_i^* = g_i, 1 \leq i \leq r-1$.

If $\lambda \in \Lambda^+(n, r)$, we define

$$\mathbf{i}_\lambda = (1^{\lambda_1}, 2^{\lambda_2}, \dots, n^{\lambda_n}) \in I(n, r), \quad (4.4)$$

where $I(n, r)$ is defined in (2.15). The following result is a special case of [25, Theorem 4.13].

Lemma 4.1. *Let V be the natural representation of $\mathbf{U}_\kappa(\mathfrak{sl}_n)$. For any $\mathfrak{t} \in \mathcal{T}^s(\lambda')$ with $\lambda \in \Lambda^+(n, r)$, let $v_{\lambda, \mathfrak{t}} = v_{\mathbf{i}_\lambda} g_{w_\lambda} \mathbf{n}_{\lambda'} g_{d(\mathfrak{t})}$. If $n \geq r$, then $\{v_{\lambda, \mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$ is a basis of κ -space consisting of all highest weight vectors of $V^{\otimes r}$ with weight $\sum_{i=1}^n \lambda_i \epsilon_i - \frac{r}{n} \sum_{i=1}^n \epsilon_i$.*

Let $\Lambda_r = \{(f, \lambda) \mid 0 \leq f \leq \lfloor r/2 \rfloor, \lambda \in \Lambda^+(r-2f)\}$. For any non-negative integer $f \leq \lfloor r/2 \rfloor$, let $\mathcal{B}_{r-2f}(\varrho, q)$ (resp., \mathcal{H}_{r-2f}) be generated by T_i and E_i (resp., g_i), $2f+1 \leq i \leq r-1$. In Theorem 4.2, $\mathbf{n}_{\mathfrak{s}\mathfrak{t}}$ is the element in $\mathcal{B}_{r-2f}(\varrho, q)$, which is obtained from that of \mathcal{H}_{r-2f} by using T_w instead of g_w .

Theorem 4.2. [12] *Let $\mathcal{B}_r(\varrho, q)$ be the Birman-Murakami-Wenzl algebra over a commutative ring R containing 1 and invertible elements ϱ, q and $q - q^{-1}$. Let*

$$S = \left\{ T_{d_1}^* E^f \mathbf{n}_{\mathfrak{s}\mathfrak{t}} T_{d_2} \mid (f, \lambda) \in \Lambda_r, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda), d_1, d_2 \in \mathcal{D}_f \right\},$$

where $E^f = E_1 E_3 \cdots E_{2f-1}$ for $f > 0$ and $E^0 = 1$.

- (1) S is a cellular basis of $\mathcal{B}_r(\varrho, q)$ over R in the sense of [13],
- (2) $\gamma(S)$ is another cellular basis of $\mathcal{B}_r(\varrho, q)$ over R , where γ is the automorphism of $\mathcal{B}_r(\varrho, q)$ defined in Lemma 2.7.

In fact, Theorem 4.2 has been given in [12] if one uses indexed representations instead of signed representations for Hecke algebras. By standard results on the representation theory on cellular algebras in [13], for each pair $(f, \lambda) \in \Lambda_r$, we have right cell modules $C(f, \lambda)$ (resp., $\tilde{C}(f, \lambda)$) of $\mathcal{B}_r(\varrho, q)$ with respect to the cellular bases of $\mathcal{B}_r(\varrho, q)$ in Theorem 4.2(1) (resp., (2)). Further, there is an invariant form $\phi_{f, \lambda}$ on $C(f, \lambda)$ (resp., $\tilde{C}(f, \lambda)$). Let $\text{rad} \phi_{f, \lambda}$ be the

radical with respect to the invariant form on $C(f, \lambda)$ (resp., $\tilde{C}(f, \lambda)$). The corresponding quotient $C(f, \lambda)/\text{rad}\phi_{f, \lambda}$ (resp., $\tilde{C}(f, \lambda)/\text{rad}\phi_{f, \lambda}$) will be denoted by $D^{f, \lambda}$ (resp., $\tilde{D}^{f, \lambda}$).

Recall that e is the order of q^2 . A partition λ of r is called e -restricted if $\lambda_i - \lambda_{i+1} < e$ for all possible i . If λ' is e -restricted, then λ is called e -regular. It is proved in [30] that $D^{f, \lambda} \neq 0$ if and only if λ is e -restricted and $f \neq r/2$ if r is even and $q^2 = 1$. By Theorem 4.2(2), similar result holds for $\tilde{D}^{f, \lambda}$. Let $P(f, \lambda)$ (resp., $\tilde{P}(f, \lambda)$) be the projective cover of $D^{f, \lambda}$ (resp., $\tilde{D}^{f, \lambda}$).

The multiplicities of simple $\mathcal{B}_r(\varrho, q)$ -modules $D^{f, \lambda}$ in cell modules $C(\ell, \mu)$ will be called decomposition numbers of $\mathcal{B}_r(\varrho, q)$ if $\varrho \neq q^{2n}$ for some $n \in \mathbb{N}$. When $\varrho = q^{2n}$, we use $\tilde{C}(\ell, \mu)$ and $\tilde{D}^{f, \lambda}$ instead of $C(\ell, \mu)$ and $D^{f, \lambda}$ to define decomposition numbers of $\mathcal{B}_r(\varrho, q)$. For any $(f, \lambda) \in \Lambda_r$, define

$$v_\lambda = \underbrace{v_1 \otimes v_{1'} \otimes \cdots \otimes v_1 \otimes v_{1'}}_{2f} \otimes v_{i_\lambda}. \quad (4.5)$$

In Proposition 4.3, we use $T_w \in \mathcal{B}_r(\varrho, q)$ instead of $g_w \in \mathcal{H}_r$ in (4.3) so as to get corresponding \mathbf{m}_λ and \mathbf{n}_λ in $\mathcal{B}_r(\varrho, q)$, where ϱ is given in (2.20).

Proposition 4.3. *Let V be the natural representation of the quantum group $\mathbf{U}_\kappa(\mathfrak{g})$ associated with $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}\}$. For any $d \in \mathcal{D}_f$ and $\mathbf{t} \in \mathcal{T}^s(\lambda')$ with $(f, \lambda) \in \Lambda_r$, define*

$$v_{\lambda, \mathbf{t}, d} = v_\lambda E^f T_{w_\lambda} \mathbf{n}_{\lambda'} T_{d(\mathbf{t})} T_d \in V^{\otimes r}.$$

If $\mathfrak{g} = \mathfrak{sp}_{2n}$ with $n \geq r$ or $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$ with $n > r$, then

- (1) *the set $\{v_{\lambda, \mathbf{t}, d} \mid \mathbf{t} \in \mathcal{T}^s(\lambda'), d \in \mathcal{D}_f\}$ is a basis of κ -space consisting of all highest weight vectors of $V^{\otimes r}$ with weight $\sum_{i=1}^n \lambda_i \epsilon_i$;*
- (2) *If $v \in V^{\otimes r}$ is a highest weight vector with weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$, then λ is a partition of $r - 2f$ for some non-negative integer f such that $(f, \lambda) \in \Lambda_r$.*

Proof. Obviously, both $v_{\lambda, \mathbf{t}, d}$ and v_λ have the same weight $\sum_{i=1}^n \lambda_i \epsilon_i$. By Corollary 2.3 and (2.14),

$$\sum_{j=0}^{2f-1} k_i^{\otimes j} \otimes e_i \otimes 1^{\otimes r-j-1} \left(v_1 \otimes v_{1'} \otimes \cdots \otimes v_1 \otimes v_{1'} \otimes v_{i_\lambda} E^f T_{w_\lambda} \mathbf{n}_{\lambda'} \right) = 0.$$

Suppose $1 \leq k \leq n$. By Lemma 2.1, e_i acts on v_k via the corresponding formulae for $\mathbf{U}_\kappa(\mathfrak{sl}_n)$ if $i \neq n$. Moreover $e_n v_k = 0$. By (2.14), $v_\lambda h = 0$ for any $h \in \mathcal{B}_r(\varrho, q)^{f+1}$. Via [12, Corollary 3.4], one can consider $T_{w_\lambda} \mathbf{n}_{\lambda'}$ in $v_\lambda E^f T_{w_\lambda} \mathbf{n}_{\lambda'}$ as the corresponding element in Hecke algebra \mathcal{H}_{r-2f} generated by $g_i, 2f+1 \leq i \leq r-1$. By Lemma 4.1, we have

$$\sum_{j=2f}^{r-1} k_i^{\otimes j} \otimes e_i \otimes 1^{\otimes r-j-1} \left(v_1 \otimes v_{1'} \otimes \cdots \otimes v_1 \otimes v_{1'} \otimes v_{i_\lambda} E^f T_{w_\lambda} \mathbf{n}_{\lambda'} \right) = 0. \quad (4.6)$$

So, $v_\lambda E^f T_{w_\lambda} \mathbf{n}_{\lambda'}$ is killed by $e_i, 1 \leq i \leq n$. In order to prove (1), it is enough to prove $\text{ann}(v_\lambda) \cap M = 0$, where

$$M = \kappa\text{-span} \{E^f T_{w_\lambda} \mathbf{n}_{\lambda'} T_{d(\mathbf{t})} T_d \mid \mathbf{t} \in \mathcal{T}^s(\lambda'), d \in \mathcal{D}_f\}.$$

If $x \in \text{ann}(v_\lambda) \cap M$, then

$$x = E^f \sum_{d \in \mathcal{D}_f} \sum_{\mathbf{t} \in \mathcal{T}^s(\lambda')} a_{\mathbf{t}} T_{w_\lambda} \mathbf{n}_{\lambda'} T_{d(\mathbf{t})} T_d$$

for some $a_t \in \kappa$ and $v_\lambda x = 0$. By arguments similar to those for Steps 1–2 in [15, Lemma 5.18], $\{v_\lambda E^f z_d T_d \mid d \in \mathcal{D}_f, z_d \neq 0\}$ is linearly independent, where $z_d = \sum_{t \in \mathcal{T}^s(\lambda')} a_t T_{w_\lambda \mathbf{n}_{\lambda'} T_d(t)}$.⁵ In particular, we have $v_{\mathbf{i}_\lambda} z_d = 0$ for any fixed d . By Lemma 4.1, $a_t = 0$ for all $t \in \mathcal{T}^s(\lambda')$ and hence $x = 0$. So, $\{v_{\lambda, t, d} \mid t \in \mathcal{T}^s(\lambda'), d \in \mathcal{D}_f\}$ is κ -linear independent.

We identify λ with $\sum_{i=1}^n \lambda_i \epsilon_i$. Let $\Delta(\lambda)$ be the Weyl module of $\mathbf{U}_\kappa(\mathfrak{g})$ with highest weight λ . Since $V^{\otimes r}$ is a tilting module, by [3, Lemma 5.1], the dimension of $\text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\Delta(\lambda), V^{\otimes r})$ is independent of κ . So, we can assume $\kappa = \mathbb{C}(v)$ and v is an indeterminate when we calculate the dimension of $\text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\Delta(\lambda), V^{\otimes r})$. In this case, $V^{\otimes r}$ is completely reducible. Since we assume $\mathfrak{g} = \mathfrak{sp}_{2n}$ with $n \geq r$ or $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$ with $n > r$, the multiplicity of irreducible $\mathbf{U}_\kappa(\mathfrak{g})$ -module L_λ (which is $\Delta(\lambda)$ in this case) is equal to the number of so-called up-down tableaux of type λ (see, e.g. [18, (5.5)]). Such a number is equal to the dimension of $C(f, \mu)$ with $\mu \in \{\lambda, \lambda'\}$ (see [12]). Thus, the cardinality of $\{v_{\lambda, t, d} \mid t \in \mathcal{T}^s(\lambda'), d \in \mathcal{D}_f\}$ is equal to the dimension of $\text{Hom}_{\mathbf{U}_\kappa}(\Delta(\lambda), V^{\otimes r})$.

Suppose $v \in V^{\otimes r}$ is a highest weight vector with weight λ . By the universal property of Weyl modules, there is an epimorphism from $\Delta(\lambda)$ to $\mathbf{U}_\kappa(\mathfrak{g})v$. It gives rise to an $f_v \in \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\Delta(\lambda), V^{\otimes r})$ sending the highest weight vector \mathbf{v} of $\Delta(\lambda)$ to v . In particular, we have $f_{\lambda, t, d}$ sending \mathbf{v} to $v_{\lambda, t, d}$ such that $\{f_{\lambda, t, d} \mid t \in \mathcal{T}^s(\lambda'), d \in \mathcal{D}_f\}$ is a basis of $\text{Hom}_{\mathbf{U}_\kappa}(\Delta(\lambda), V^{\otimes r})$. This implies (1).

If there is a highest weight vector $v \in V^{\otimes r}$ with weight μ , then there is an epimorphism from $\Delta(\mu)$ to $\mathbf{U}_\kappa v$ and hence $\dim \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\Delta(\mu), V^{\otimes r}) \neq 0$. Since $V^{\otimes r}$ is a tilting module, by [3, Lemma 5.1], such a dimension is independent of κ . So, we assume $\kappa = \mathbb{C}(v)$. In this case, $V^{\otimes r}$ is completely reducible. By [18, (5.5)], $\mu = \sum_{i=1}^n \mu_i \epsilon_i$ such that $(f, \mu) \in \Lambda_r$ and (2) follows. \square

Abusing of notation, we denote $\sum_{i=1}^n \lambda_i \epsilon_i$ by λ . In the remaining part of this section, we denote by $\nabla(\lambda)$ the co-Weyl module of $\mathbf{U}_\kappa(\mathfrak{g})$ with respect to the highest weight λ .

We always keep assumptions that either $\mathfrak{g} = \mathfrak{sp}_{2n}$ with $n \geq r$ or $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$ with $n > r$. Let V be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$. For any $\mathbf{U}_\kappa(\mathfrak{g})$ -module M , $\text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(M, V^{\otimes r})$ is a right $\mathcal{B}_r(\varrho, q)$ -module in the sense

$$(\varphi b)(x) = \varphi(x) \alpha(b), \quad (4.7)$$

for all $x \in M$, $b \in \mathcal{B}_r(\varrho, q)$, and $\varphi \in \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(M, V^{\otimes r})$ where α is the automorphism γ (resp., identity automorphism) given in Lemma 2.7(2) if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ (resp., $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{sp}_{2n}\}$). We remark that $\text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(M, V^{\otimes r})$ can be considered as a left $\mathcal{B}_r(\varrho, q)$ -module such that $xf := f\sigma(x)$, $\forall x \in \mathcal{B}_r(\varrho, q)$ and $f \in \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(M, V^{\otimes r})$, where σ is the anti-involution on $\mathcal{B}_r(\varrho, q)$ defined in Lemma 2.7. Let $\mathbf{U}_\kappa(\mathfrak{g})\text{-mod}$ (resp., $\text{mod-}\mathcal{B}_r(\varrho, q)$) be the category of finite dimensional left $\mathbf{U}_\kappa(\mathfrak{g})$ -modules (resp., right $\mathcal{B}_r(\varrho, q)$ -modules) over κ . Later on, we define

$$\mathcal{F} = \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(-, V^{\otimes r}). \quad (4.8)$$

Proposition 4.4. *Suppose $(f, \lambda) \in \Lambda_r$.*

- (1) *If $\mathfrak{g} = \mathfrak{sp}_{2n}$ with $n \geq r$, then $\mathcal{F}(\Delta(\lambda)) \cong C(f, \lambda')$ as right $\mathcal{B}_r(\varrho, q)$ -modules.*
- (2) *If $\mathfrak{g} = \mathfrak{so}_{2n}$ with $n > r$, then $\mathcal{F}(\Delta(\lambda)) \cong C(f, \lambda')$ as right $\mathcal{B}_r(\varrho, q)$ -modules.*

⁵Although Hu proves the result for $\mathbf{U}_\kappa(\mathfrak{sp}_{2n})$, his arguments can be used smoothly for both $\mathbf{U}_\kappa(\mathfrak{so}_{2n})$ and $\mathbf{U}_\kappa(\mathfrak{so}_{2n+1})$. The key point is that Hu's arguments depend on [15, (5.13)] and does not depend on the explicit formulae for $(v_k \otimes v_l)\check{R}$. So, we can use Corollary 2.5 instead of [15, (5.13)].

(3) If $\mathfrak{g} = \mathfrak{so}_{2n+1}$ with $n > r$, then $\mathcal{F}(\Delta(\lambda)) \cong \tilde{C}(f, \lambda')$ as right $\mathcal{B}_r(\varrho, q)$ -modules.

Proof. It is routine to prove that

$$C(f, \lambda') \cong E^f \mathfrak{m}_\lambda T_{w_\lambda} \mathfrak{n}_{\lambda'} \mathcal{B}_r(\varrho, q) \pmod{\mathcal{B}_r(\varrho, q)^{f+1}},$$

where $\mathcal{B}_r(\varrho, q)^{f+1}$ is the two-sided ideal of $\mathcal{B}_r(\varrho, q)$ generated by E^{f+1} . Further, as a κ -space, $E^f \mathfrak{m}_\lambda T_{w_\lambda} \mathfrak{n}_{\lambda'} \mathcal{B}_r(\varrho, q) \pmod{\mathcal{B}_r(\varrho, q)^{f+1}}$ has a κ -basis consisting of $E^f \mathfrak{m}_\lambda T_{w_\lambda} \mathfrak{n}_{\lambda'} T_{d(t)} T_d \pmod{\mathcal{B}_r(\varrho, q)^{f+1}}$, for all $(t, d) \in \mathcal{T}^s(\lambda') \times \mathcal{D}_f$. Let $f_{\lambda, t, d} \in \mathcal{F}(\Delta(\lambda))$ sending the highest weight vector of $\Delta(\lambda)$ to $v_{\lambda, t, d} \in V^{\otimes r}$ in Proposition 4.3. Then $f_{\lambda, t, d}$'s are κ -base elements of $\mathcal{F}(\Delta(\lambda))$. It is routine to check that the required isomorphism Φ in (1) is the κ -linear isomorphism satisfying

$$\Phi(f_{\lambda, t, d}) = E^f \mathfrak{m}_\lambda T_{w_\lambda} \mathfrak{n}_{\lambda'} T_{d(t)} T_d + \mathcal{B}_r(\varrho, q)^{f+1}.$$

Finally, (2)–(3) can be proved similarly. The reason why we use right cell module $\tilde{C}(f, \lambda')$ in (3) is that we use usual linear dual in Corollary 3.2 when $\mathfrak{g} = \mathfrak{so}_{2n+1}$. \square

For each left $\mathbf{U}_\kappa(\mathfrak{g})$ -module M , $\text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(V^{\otimes r}, M)$ is a left $\mathcal{B}_r(\varrho, q)$ -module such that, for any $x \in V^{\otimes r}$, $b \in \mathcal{B}_r(\varrho, q)$ and $\phi \in \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(V^{\otimes r}, M)$,

$$(b\phi)(x) = \phi(xb). \quad (4.9)$$

Also, $V^{\otimes r} \otimes_{\mathcal{B}_r(\varrho, q)} N$ is a left $\mathbf{U}_\kappa(\mathfrak{g})$ -module for any left $\mathcal{B}_r(\varrho, q)$ -module N . In the following, let $\mathcal{B}_r(\varrho, q)\text{-mod}$ be the category of left $\mathcal{B}_r(\varrho, q)$ -modules.

Definition 4.5. Let \mathbf{f} and \mathbf{g} be two functors

$$\begin{aligned} \mathbf{f} : \mathbf{U}_\kappa(\mathfrak{g})\text{-mod} &\longrightarrow \mathcal{B}_r(\varrho, q)\text{-mod} \\ M &\longmapsto \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(V^{\otimes r}, M), \\ \mathbf{g} : \mathcal{B}_r(\varrho, q)\text{-mod} &\longrightarrow \mathbf{U}_\kappa(\mathfrak{g})\text{-mod} \\ N &\longmapsto V^{\otimes r} \otimes_{\mathcal{B}_r(\varrho, q)} N. \end{aligned}$$

It follows from [16, Theorem 2.11] that \mathbf{f} and \mathbf{g} are adjoint pairs in the sense that

$$\text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\mathbf{g}(N), M) \cong \text{Hom}_{\mathcal{B}_r(\varrho, q)}(N, \mathbf{f}(M)), \quad (4.10)$$

as κ -spaces where M (resp., N) is a left $\mathbf{U}_\kappa(\mathfrak{g})$ -module (resp., left $\mathcal{B}_r(\varrho, q)$ -module N).

Lemma 4.6. Let T be an indecomposable direct summand of the left $\mathbf{U}_\kappa(\mathfrak{g})$ -module $V^{\otimes r}$. Then $\mathbf{g}\mathbf{f}(T) \cong T$.

Proof. By Theorem 2.15, $\mathbf{f}(V^{\otimes r}) \cong \mathcal{B}_r(\varrho, q)$ and hence $\mathbf{g}\mathbf{f}(V^{\otimes r}) \cong V^{\otimes r}$. The corresponding isomorphism ϕ sends $v \otimes b$ to $b(v)$ for any $v \in V^{\otimes r}$ and $b \in \mathbf{f}(V^{\otimes r})$. Since T is a direct summand of $V^{\otimes r}$, the projection $\pi : V^{\otimes r} \rightarrow T$ induces a homomorphism $1 \otimes \mathbf{f}(\pi)$ from $\mathbf{g}\mathbf{f}(V^{\otimes r})$ to $\mathbf{g}\mathbf{f}(T)$ such that $\pi \circ \phi = \tilde{\phi} \circ (1 \otimes \mathbf{f}(\pi))$ where $\tilde{\phi}$ is the homomorphism from $\mathbf{g}\mathbf{f}(T)$ to T sending $v \otimes h$ to $h(v)$ where $v \in V^{\otimes r}$ and $h \in \mathbf{f}(T)$. So, $\tilde{\phi}$ is surjective. Comparing dimensions yields that $\tilde{\phi}$ is an isomorphism. \square

We remark that any right $\mathcal{B}_r(\varrho, q)$ -module can be considered as left $\mathcal{B}_r(\varrho, q)$ -module via the anti-involution σ in Lemma 2.7 and vice versa. In Theorem 4.7, let ω_0 be the longest element of the Weyl group associated to \mathfrak{g} . In the remaining part of this paper, let $T(\lambda)$ be the indecomposable (or partial) tilting module of $\mathbf{U}_\kappa(\mathfrak{g})$ with respect to the highest weight

λ . Let $(T(\lambda) : \Delta(\mu))$ be the multiplicity of the Weyl module $\Delta(\mu)$ in $T(\lambda)$. This multiplicity is well-defined since it is independent of Weyl filtrations of $T(\lambda)$.

Theorem 4.7. *Let V be the natural representation of $\mathbf{U}_\kappa(\mathfrak{g})$ such that $\mathfrak{g} = \mathfrak{sp}_{2n}$ (resp., $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}\}$) with $n \geq r$ (resp., $n > r$). Let $\mathcal{B}_r(\varrho, q)$ be the Birman-Murakami-Wenzl algebra over κ , where ϱ is given in (2.20).*

- (1) *Any partial tilting module which appears as an indecomposable direct summand of $V^{\otimes r}$ is of form $T(\lambda)$ for some $(f, \lambda') \in \Lambda_r$ with λ being e -regular.*
- (2) *Suppose $\mathfrak{g} = \mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$. As right $\mathcal{B}_r(\varrho, q)$ -modules,*
 - (a) $\mathbf{f}(\nabla(\mu)) \cong \mathcal{F}(\Delta(\mu)) \cong C(f, \mu')$ for any $(\ell, \mu) \in \Lambda_r$,
 - (b) $\mathbf{f}(T(\lambda)) \cong P(f, \lambda')$ for any $(f, \lambda') \in \Lambda_r$ with λ being e -regular.
 - (c) $(T(\lambda) : \Delta(\mu)) = [C(\ell, \mu') : D^{f, \lambda'}]$.
- (3) *Suppose $\mathfrak{g} = \mathfrak{so}_{2n+1}$. As right $\mathcal{B}_r(\varrho, q)$ -modules,*
 - (a) $\mathbf{f}(\nabla(-\omega_0\mu)) \cong \mathcal{F}(\Delta(\mu)) \cong \tilde{C}(f, \mu')$, for $(\ell, \mu) \in \Lambda_r$,
 - (b) $\mathbf{f}(T(-\omega_0\lambda)) \cong \tilde{P}(f, \lambda')$ for any $(f, \lambda') \in \Lambda_r$ with λ being e -regular.
 - (c) $(T(-\omega_0\lambda) : \Delta(-\omega_0\mu)) = [\tilde{C}(\ell, \mu') : \tilde{D}^{f, \lambda'}]$.

Proof. Suppose $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{sp}_{2n}\}$. Let $\Psi : \mathcal{F}(\Delta(\mu)) \rightarrow \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}((V^{\otimes r})^\circ, \Delta(\mu)^\circ)$ be the κ -linear isomorphism such that

$$[\Psi(\phi)(v^\circ)](x) = \langle v, \phi(x) \rangle = v^\circ(\phi(x)), \forall \phi \in \mathcal{F}(\Delta(\mu)), v \in V^{\otimes r}, x \in \Delta(\mu) \quad (4.11)$$

where $\langle \cdot, \cdot \rangle$ is the bilinear form defined in (3.11) and $v^\circ \in \text{Hom}_\kappa(V^{\otimes r}, \kappa)$ is defined in (3.20). So, for any $b \in \mathcal{B}_r(\varrho, q)$,

$$\begin{aligned} (\Psi(b\phi)(v^\circ))(x) &= \langle v, b\phi(x) \rangle, \text{ by (4.11),} \\ &= \langle v, \phi(x)\sigma(b) \rangle \text{ by (4.7),} \\ &= (v^\circ b)(\phi(x)) \text{ by (3.10) and (3.20),} \\ &= [\Psi(\phi)(v^\circ b)](x), \text{ by (4.11).} \end{aligned} \quad (4.12)$$

and hence by (4.9), $\Psi(b\phi)(v^\circ) = \Psi(\phi)(v^\circ b) = (b(\Psi(\phi)))(v^\circ)$. So $\Psi(b\phi) = b\Psi(\phi)$. By Corollary 3.4, and $\nabla(\mu) \cong \Delta(\mu)^\circ$ for any dominant integral weight μ (see [10, Proposition 4.1.6]), $\mathbf{f}(\nabla(\mu)) \cong \mathcal{F}(\Delta(\mu))$ as left $\mathcal{B}_r(\varrho, q)$ -modules. Via anti-involution, it can be considered as isomorphism for right $\mathcal{B}_r(\varrho, q)$ -modules. Finally, the last isomorphism in 2(a) follows from Proposition 4.4.

By [5, II, Proposition 2.1(c)], the functor \mathbf{f} in Definition 4.5 induces a category equivalence between direct sums of direct summands of the left $\mathbf{U}_\kappa(\mathfrak{g})$ -module $V^{\otimes r}$ and direct sums of direct summands of left $\mathcal{B}_r(\varrho, q)$ -module $\mathcal{B}_r(\varrho, q)$. So $\mathbf{f}(T(\mu))$ is a principal indecomposable left $\mathcal{B}_r(\varrho, q)$ -module if $T(\mu)$ is an indecomposable direct summand of $V^{\otimes r}$. By the universal property, the Weyl module $\Delta(\mu)$ is a submodule of $T(\mu)$. So, $\dim \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\Delta(\mu), V^{\otimes r}) \neq 0$. By arguments in the proof of Proposition 4.3(2), μ has to be a partition such that $(\ell, \mu) \in \Lambda_r$ for some ℓ . So, $(\ell, \mu') \in \Lambda_r$. For any $(k, \nu') \in \Lambda_r$, we have κ -linear isomorphism

$$\begin{aligned} \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(T(\mu), \nabla(\nu)) &\cong \text{Hom}_{\mathbf{U}_\kappa(\mathfrak{g})}(\mathbf{g}\mathbf{f}(T(\mu)), \nabla(\nu)), \text{ by Lemma 4.6,} \\ &\cong \text{Hom}_{\mathcal{B}_r(\varrho, q)}(\mathbf{f}(T(\mu)), \mathbf{f}(\nabla(\nu))), \text{ by (4.10),} \\ &\cong \text{Hom}_{\mathcal{B}_r(\varrho, q)}(P(f, \lambda'), C(k, \nu')), \text{ by 2(a),} \end{aligned} \quad (4.13)$$

for some $(f, \lambda') \in \Lambda_r$ such that $\mathbf{f}(T(\mu)) = P(f, \lambda')$ with λ being e -regular. Note that $P(f, \lambda')$ is a principal indecomposable module, we have

$$\dim_{\kappa} \operatorname{Hom}_{\mathcal{B}_r(\varrho, q)}(P(f, \lambda'), C(k, \nu')) = [C(k, \nu') : D^{f, \lambda'}]. \quad (4.14)$$

If we assume $\nu = \mu$, then $(\ell, \mu') \supseteq (f, \lambda')$. If we assume $\nu = \lambda$, then $\operatorname{Hom}_{\mathbf{U}_{\kappa}(\mathfrak{g})}(T(\mu), \nabla(\lambda)) \neq 0$. Since μ is the highest weight of $T(\mu)$, $\lambda \leq \mu$ and either $f > \ell$ or $f = \ell$, forcing $(\ell, \mu') \supseteq (f, \lambda')$. So, $f = \ell$ and $\mu = \lambda$. This proves 2(b) as left $\mathcal{B}_r(\varrho, q)$ -modules. Via anti-involution σ in Lemma 2.7, we have 2(b) as right $\mathcal{B}_r(\varrho, q)$ -modules. In particular, we have proved (1) for $\mathfrak{g} \in \{\mathfrak{so}_{2n}, \mathfrak{sp}_{2n}\}$. Finally, 2(c) follows from (4.13)-(4.14) and $(T(\lambda) : \Delta(\mu)) = \dim_{\kappa} \operatorname{Hom}_{\mathbf{U}_{\kappa}(\mathfrak{g})}(T(\mu), \nabla(\nu))$.

Suppose $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $\Psi : \mathcal{F}(\Delta(\mu)) \rightarrow \operatorname{Hom}_{\mathbf{U}_{\kappa}(\mathfrak{g})}((V^{\otimes r})^*, \Delta(\mu)^*)$ be the κ -linear isomorphism such that

$$\Psi(\phi)(v^*)(x) = \langle v, \phi(x) \rangle, \forall \phi \in \mathcal{F}(\Delta(\mu)), v \in V^{\otimes r}, x \in \Delta(\mu), \quad (4.15)$$

where $\langle \cdot, \cdot \rangle$ is defined (3.1) and v^* is defined in a natural way. By arguments similar to those for 2(a)-(b), one can check that Ψ is a left $\mathcal{B}_r(\varrho, q)$ -isomorphism. So, 3(a) follows from Corollary 3.2, Proposition 4.4 and the fact $\nabla(-\omega_0 \mu) \cong \Delta(\mu)^*$ (see [2, Proposition 3.3]). Finally, 3(b)-(c) and (1) for \mathfrak{so}_{2n+1} follow from arguments similar to those for 2(b) and (1) for \mathfrak{so}_{2n} and \mathfrak{sp}_{2n} . \square

We close the paper by giving the following remarks on decomposition numbers of $\mathcal{B}_r(\varrho, q)$ over \mathbb{C} .

Remark 4.8. (1) Suppose that $\varrho \notin \{q^a, -q^a \mid a \in \mathbb{Z}\}$. In [24], Rui and Si have proved that $\mathcal{B}_r(\varrho, q)$ is Morita equivalent to $\bigoplus_{i=0}^{\lfloor r/2 \rfloor} \mathcal{H}_{r-2i}$ over κ .

- (a) If q^2 is not a root of unity, $\mathcal{B}_r(\varrho, q)$ is split semisimple⁶ and the decomposition matrix of $\mathcal{B}_r(\varrho, q)$ is the identity matrix.
 - (b) If q^2 is a root of unity and κ is \mathbb{C} , by Ariki's famous results on LLT conjecture in [4], decomposition numbers of $\mathcal{B}_r(\varrho, q)$ are determined by values of certain inverse Kazhdan-Lusztig polynomials associated with some extended affine Weyl groups of type A at $q = 1$. In this case, there is no restriction on e , the order of q^2 .
- (2) Suppose $\varrho \in \{-q^a, q^a \mid a \in \mathbb{Z}\}$.
- (a) If q^2 is not a root of unity, Xu showed that $\mathcal{B}_r(\varrho, q)$ is multiplicity free over \mathbb{C} [31].
 - (b) If $\kappa = \mathbb{C}$ and $o(q^2) = e$, we assume $q = \exp(2\pi i/e)$ if e is odd and $q = \exp(\pi i/e)$ if e is even. Further, we assume that $q^{1/2} = \exp(\pi i/2e)$ if e is even. In this case, $q^{1/2}$ is a primitive $4e$ -th root of unity. If e is odd, $-q^{2n} \in \{-q^{2k+1} \mid k \in \mathbb{Z}\}$. If e is even, $q^e = -1$ and $-q^{2n} = q^{2n+e}$. Finally, if $\varrho = q^{2n}$ and e is odd, $\varrho = q^{2n+e}$. In summary, when we calculate decomposition numbers of $\mathcal{B}_r(\varrho, q)$ for $\varrho \in \{-q^a, q^a \mid a \in \mathbb{Z}\}$ and q^2 being a root of unity, we can always assume that ϱ 's are given in (2.20). Moreover, we can assume e is even if $\varrho = q^{2n}$ for some $n \in \mathbb{Z}$. By Theorem 4.7, decomposition numbers of $\mathcal{B}_r(\varrho, q)$ are determined

⁶ A necessary and sufficient condition on the semisimplicity of $\mathcal{B}_r(\varrho, q)$ has been given in [23]. See also [28] for some partial results over \mathbb{C} .

by multiplicities of Weyl modules in certain indecomposable tilting modules for $U_\kappa(\mathfrak{g})$. Soergel [26] has described multiplicities of Weyl modules in certain indecomposable tilting modules for $U_\kappa(\mathfrak{g})$ via the equivalence of categories between modules for quantum groups at roots of unity and corresponding module categories for Kac-Moody algebras in [27]. Due to [17], this equivalence is only proved when $e \geq 29$. In principal, we know decomposition numbers of $\mathcal{B}_r(\varrho, q)$ for $\varrho \in \{-q^a, q^a \mid a \in \mathbb{Z}\}$ over \mathbb{C} only if $e \geq 29$.

REFERENCES

- [1] H.H ANDERSEN “Tensor products of quantized tilting modules ”, *Comm. Math. Physics.*, **149**, (1992), 149–159.
- [2] H.H ANDERSEN, P. POLO AND K.X. WEN “Representations of quantum algebras”, *Invent. Math.*, **104**, (1991), 1–59.
- [3] H.H ANDERSEN, G. LEHRER AND R. ZHANG “Cellularity of certain quantum endomorphism algebras”, *preprint*, arXiv:1303.0984v1.
- [4] S. ARIKI, “On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$ ” *J. Math. Kyoto Univ.* **36** (1996), no. 4, 789–808.
- [5] M. AUSLANDER, I. REITEN, S. SMALO, “Representation theory of Artin algebras ”, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, 1995.
- [6] J. S. BIRMAN and H. WENZL, “Braids, link polynomials and a new algebra”, *Trans. Amer. Math. Soc.* **313** (1989), 249–273.
- [7] N. BOURBAKI, “Elements of Mathematics, Lie Groups and Lie Algebras, Chapters 4–6” *Springer*, (2002).
- [8] CHRISTIAN KASSEL , “Quantum Group”, *Graduate Texts in Mathematics* , **155**.
- [9] R. DIPPER, S. DOTY, J. HU, “Brauer algebras, symplectic Schur algebras and Schur-Weyl duality ”, *Trans. Amer. Math. Soc.*, **360**(2008), 189–213.
- [10] S. DONKIN, “The q -Schur algebra ”, *London Mathematical Society Lecture Note Series*, **253**, Cambridge University Press, 1998.
- [11] S. DONKIN AND R. TANGE, “The Brauer algebra and the symplectic Schur algebra *Math. Z.* **265** (2010), 187–219.
- [12] J. ENYANG, “Specht modules and semisimplicity criteria for Brauer and Birman-Murakami-Wenzl algebras”, *J. Algebraic Combin.* **26** (2007), no. 3, 291–341.
- [13] J. J. GRAHAM and G. I. LEHRER, “Cellular algebras”, *Invent. Math.* **123** (1996), 1–34.
- [14] T. HAYASHI, “Quantum deformation of Classical Groups”, *Publ. RIMS, Kyoto Univ.* **28** (1992), 57–81.
- [15] J. HU “BMW algebra, quantized coordinate algebra and type C Schur-Weyl duality” *Represent. Theory* **15** (2011), 1–62.
- [16] J. J. ROTMAN, “An introduction to homological algebra ”, *Pure and Applied Mathematics*, **85**, Academic Press, New York-London, 1979.
- [17] D. KAZHDAN AND G. LUSZTIG “Tensor structures arising from affine Lie algebras. I–IV”, *J. Amer. Math. Soc.* **6** (1993), 905–947, 949–1011, **7** (1994), 335–381 and 383–453.
- [18] R. LEDUC AND A. RAM “A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman-Wenzl, and type A Iwahori-Hecke algebras”, *Adv. Math.* **125** (1997), no. 1, 1–94.
- [19] P. MARTIN “The decomposition matrices of the Brauer algebra over the complex field”, *preprint*.
- [20] J. MURAKAMI “The Kauffman polynomial of links and representation theory” *Osaka Journal of Mathematics* **24** (1987), 745–758.
- [21] G. LUSZTIG, “Quantum groups at roots of 1, *Geom. Dedicata* **35** (1990), 89–113.
- [22] H. RUI and M. SI “Blocks of Birman-Murakami-Wenzl algebras”, *Int. Math. Res. Not.*, 2011, no. 2, 452–486.
- [23] H. RUI and M. SI “Gram determinants and semisimplicity criteria for Birman-Wenzl algebras”, *J. Reine Angew. Math.* **631** (2009), 153–179.
- [24] H. RUI and M. SI “Singular parameters for the Birman-Murakami-Wenzl algebra”, *J. Pure Appl. Algebra*, **216** (2012), no. 6, 1295–1305.
- [25] H. RUI and L. SONG, “Decomposition numbers of quantized walled Brauer algebras”, *preprint*.

- [26] W. SOERGEL “Kazhdan-Lusztig polynomials and a combinatorics for tilting modules”, *Represent. Theory* **1** (1997), 83–114.
- [27] W. SOERGEL “Character formulas for tilting modules over Kac-Moody algebras”, *Represent. Theory* **2** (1998), 432–448
- [28] H. WENZL “Quantum groups and subfactors of type B, C, and D.” *Comm. Math. Phys.* **133** (1990), no. 2, 383–432.
- [29] H. WENZL “Quotients of Representation Rings ” *Represent. Theory* **15** (2011), 385–406
- [30] C.C. XI , “On the quasi-heredity of Birman-Wenzl algebras, ”, *Adv. Math.* **154**(2) (2000), 280-298.
- [31] X. XU “Decomposition numbers of cyclotomic NW and BMW algebras.” *J. Pure Appl. Algebra*, **217** (2013), no. 6, 1037–1053.

H. RUI: DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, 200241, CHINA
E-mail address: `hbrui@math.ecnu.edu.cn`

L. SONG: MATHEMATICS AND SCIENCE COLLEGE, SHANGHAI NORMAL UNIVERSITY, SHANGHAI, 200234, CHINA
E-mail address: `song51090601020@163.com`